TECHNISCHE UNIVERSITÄT DARMSTADT

FACHBEREICH MATHEMATIK

Bachelor's thesis

Equivariant fundamental groupoids of C_2-surfaces

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December 19, 2025

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Contents

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Introduction

In this thesis we calculate the equivariant fundamental groupoid of several C_2-surfaces.

The non-equivariant fundamental groupoid of a topological space X, denoted by $\Pi(X)$, is a generalization of the fundamental group $\pi_1(X, x_0)$ which is independent of the base point x_0. Its elements are path homotopy classes between any two points, and composition is only defined when the first path ends at the starting point of the second path.

Let C_2 denote the group of order 2. By a C_2-surface we mean a connected, closed surface together with a continuous action of C_2. For example, there are five non-trivial C_2-tori, illustrated here as reflections at the blue set in $\mathbb{R}^3$:

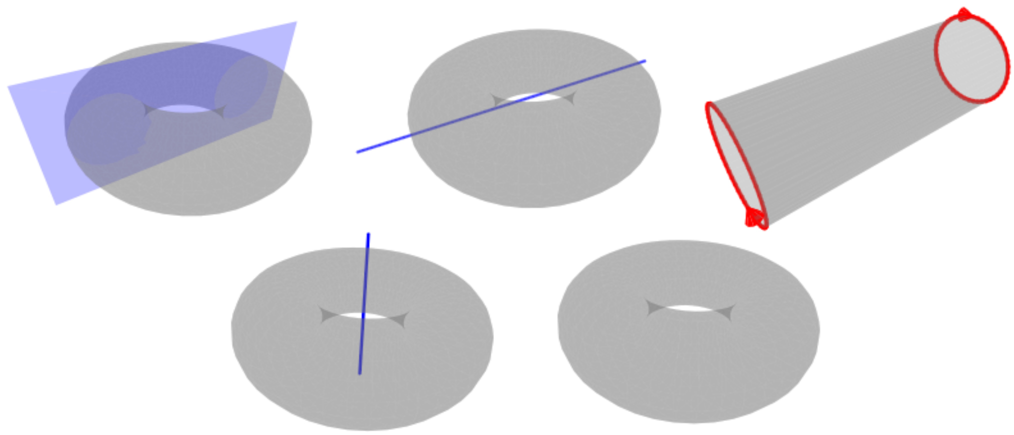


Figure 1: Non-trivial C_2-tori

The full classification of C_2-surfaces is due to Dugger [Dug16].

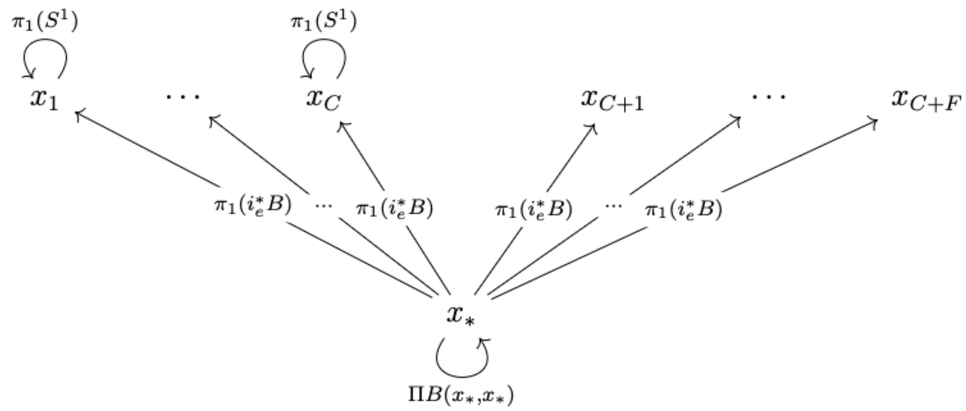
The equivariant fundamental groupoid ΠB of a C_2-surface B is a generalization of the fundamental groupoid of the underlying surface $i_e^* B$. It captures information about the fixed set B^{C_2} and the C_2-action in general. It can be described via a skeleton of it, which can be thought of as fixing a set of base points.

The equivariant fundamental groupoid is a C_2-homotopy invariant, in the sense that the equivariant fundamental groupoids of C_2-homotopic C_2-spaces are equivalent. In the non-equivariant setting, the fundamental groupoid classifies the closed surfaces. This raises the question of how strong an invariant it is for the case of C_2-surfaces:
Question (1.36)**.** *Is a C_2-surface uniquely determined by its equivariant fundamental groupoid and the fundamental groupoid of its underlying surface?*

This thesis answers this question for the C_2-spheres and the C_2-tori:
Theorem (2.1, 2.2)**.** *Let B be a C_2-sphere or a C_2-torus. Then ΠB determines which one.*

However, we show that in general the equivariant fundamental groupoid alone is insufficient:
Theorem. *The torus with its antipodal C_2-action and the Klein bottle with its unique free C_2-action have equivalent equivariant fundamental groupoids.*

Furthermore, this thesis presents general results regarding the equivariant fundamental groupoid of a C_2-surface:
Theorem (1.31)**.** *Let B be a C_2-surface with non-trivial action. Then ΠB has the following skeleton, where C is the number of fixed circles and F is the number of fixed points:*



Further, there is a short exact sequence

$$1 \longrightarrow \pi_1(i_e^*B) \xrightarrow{i} \Pi B(x_*,x_*) \xrightarrow{\rho} C_2 \longrightarrow 1$$

Calculating the equivariant fundamental groupoid of a C_2-surface is difficult because it is hard to find the group $\Pi B(x_*,x_*)$. Nevertheless, we show the following:

Theorem (1.33)**.** *Let B be a C_2-surface with non-trivial action. The above short exact sequence is right split if and only if B has fixed points or $B \cong S^2_a$ is the sphere with the antipodal action.*

The sequence is right split if and only if $\Pi B(x_*, x_*)$ is a semidirect product of the form $\pi_1(i_e^* B) \rtimes_\varphi C_2$. This is due to a version of the splitting lemma for general groups. This raises another question:
Question (1.37)**.** *Let X be a closed surface. Do all possible semidirect products of the form $\pi_1(X) \rtimes_\varphi C_2$ occur as $\Pi B(x_*, x_*)$ of some C_2-surface B with $i_e^* B = X$?*

This turns out to be true for both the C_2-spheres and the C_2-tori. In particular, there is an intuitive bijection between the similarity classes of self-inverse 2×2 integer matrices and C_2-tori with fixed points (see Theorem 2.3).

The thesis is structured as follows. In Chapter 1 we define the equivariant fundamental groupoid for G-spaces where G is a finite group. We also sketch the classification of closed surfaces, calculate their fundamental groups, state the classification of orientable C_2-surfaces, and finally, prove the Theorem 1.31 and Theorem 1.33.

Chapter 2 determines the equivariant fundamental groupoid for all four C_2-spheres and all six C_2-tori in detail, finding that the answers to the questions above is "yes" in these cases. We also discuss the free C_2-action on the Klein bottle and show that the equivariant fundamental groupoid of a C_2-surface does not determine the underlying surface.

Finally, we determine the equivariant fundamental groupoid for the class of C_2-surfaces denoted by $T^{\text{refl}}_{g,r}$. These are orientable C_2-surfaces where the action is best described as a reflection along a plane in $\mathbb{R}^3$:

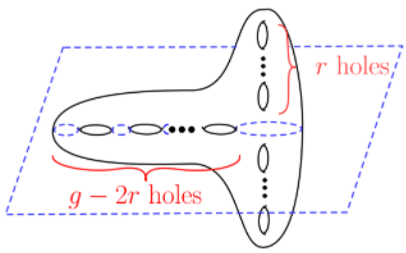


Figure 2: $T^{\text{refl}}_{g,r}$, [Dug16, 5.1]

Acknowledgements

First of all, I'd like to thank my supervisor, Prof. Dr. Sabrina Pauli, for our lengthy meetings and her regular feedback, which both were very helpful and guided me through this thesis, as well as for her awesome previous lecture on algebraic topology, which won me for the topic.

Thanks also to Prof. Dr. Karsten Große-Brauckmann for lending me books of his (even though they couldn't solve my problem at hand at the time) and for teaching the geometric view on manifolds in their lecture on Riemannian geometry.

I also want to thank my friends Justus Kempfer, Franziska Lach, and Lara Tschendel for their interest in my thesis and being very helpful rubber-ducks.

Further, I want to thank my dad for being able to use his workshop to build a wood model of orientable surfaces, as this greatly helped with gaining intuition.

Thanks for partial proof reading to my mum and to Justus.

Thanks also to Wikipedia and everyone contributing to its math content, which I used widely to look up context or theorems.

I also want to thank for all the tools I used to create the images, including Java with Processing and Peasycam, as well as Geogebra.

Chapter 1

Foundations

1.1 The equivariant fundamental groupoid

In this section we will start with the basic definitions needed to define the equivariant fundamental groupoid for topological G-spaces with a finite group G. In the end of this section we compute the fundamental groupoid of $S^{1,1}$, the circle with reflection along a diameter, to introduce the process of calculating equivariant fundamental groupoids of G-spaces and to establish colours and variable names used throughout this thesis.

The reader is assumed to have some introductory course level knowledge about topology, groups and categories.

By G we always denote a finite group.

Definition 1.1 (G-space, [Die87]). *A G-space (X, ϕ) is a topological space X together with a left group action $\phi\colon G \times X \to X$ which acts continuously, i.e. for all $g \in G$ the map $\phi(g, \cdot)\colon X \to X$ is continuous.*

When (X, ϕ) is a G-space and $g \in G$, we also denote the map $\phi(g, \cdot)\colon X \to X$ by g. Since g is invertible, for all $g \in G$ this map is continuous and continuously invertible, i.e. a homeomorphism from X to X.

We will focus on the case where $G = C_2$ is the cyclic group of two elements. Then the action of the non-identity element of C_2 on X is an *involution* τ on X, i.e. a continuous map $\tau\colon X \to X$ with $\tau^2 = id$. So a C_2-space can also be understood as a topological space together with an involution τ. [Dug16, Introduction].

We will always denote the elements of C_2 by e, the identity element, and τ, the non-identity element.

Definition 1.2 (G-map)**.** *Let (X_1, ϕ_1), (X_2, ϕ_2) be two G-spaces. Then a G-*map *from (X_1, ϕ_1)* to *(X_2, ϕ_2) is a continuous map $f : X_1 \to X_2$ which commutes with the action of g, i.e. $f(\phi_1(g, x)) = \phi_2(g, f(x))$, or with the usual short notation for group actions $f(gx) = gf(x)$. G-maps are also called* equivariant maps*.*

Definition 1.3 (G-homotopy)**.** *Let $(X_1, \phi_1), (X_2, \phi_2)$ be G-spaces, and let $f, g \colon X_1 \to X_2$ be G-maps. Then a G-*homotopy from *f* to *g is a homotopy $h \colon X_1 \times [0, 1] \to X_2$ from f to g which is also a G-map from $(X_1 \times [0, 1], \phi_1 \times \mathrm{id})$ to (X_2, ϕ_2), or equivalently such that for all $t \in [0, 1]$ the map $h(\cdot, t) : X_1 \to X_2$ is a G-map.*

Remark 1.4. Let G be a finite group. One can consider G as a topological space with the discrete topology, turning G into a topological group. Then all of our above definitions are compatible with the general case of G being any topological group.

Remark 1.5. For any subgroup H in G, this concept turns G/H into a G-space where the action is left multiplication.

[Die87, Example 1.4]

Definition 1.6 (Top_G)**.** *Let* Top_G *denote the category of G-spaces. Morphisms in this category are G-maps and composition is composition of functions.*

Let (X, ϕ) be a G-space, H a finite group and $\psi : H \to G$ a group homomorphism. Then H acts on X continuously via $\phi(\psi(\cdot), \cdot) \colon H \times X \to X$, turning X into a H-space $(X, \phi(\psi(\cdot), \cdot))$. So ψ defines a functor $\psi^* : \mathrm{Top}_G \to \mathrm{Top}_H$.

If H is a subgroup of G we write $i_H \colon H \hookrightarrow G$ for the inclusion and then i_H^* is called the *restriction functor*. It restricts the action of the G-space to the subgroup.

For the trivial group with one element e every continuous map commutes with the group action which is just the identity on any topological space. So $\mathrm{Top}_e \cong \mathrm{Top}$.

Given a G-space $B = (X, \phi)$, this leads to the common notation of $i_e^* B = X$ to denote the underlying topological space of a G-Space B. Here i_e is the inclusion $i_e \colon \{e\} \to G$. [BLM+24]

By B we will always denote a G-space.

We are ready to define the equivariant fundamental groupoid of a G-space B. First we give the full definition of the equivariant fundamental groupoid and then we give intuition about its meaning afterwards. The order of composition of functions is as usual and the concatenation of homotopies ($\star$) is in the same order as composition of functions. By I we denote the unit interval $[0, 1]$ with trivial G-action.

Definition 1.7 (Equivariant fundamental groupoid [Die87]). *The* equivariant fundamental groupoid of a G-space B *is the category* ΠB *with*

- *objects being G-maps $x\colon G/H \to B$, where H is any subgroup of G*
- *morphisms from $x\colon G/H \to B$ to $y\colon G/K \to B$ being pairs $(\alpha, [w])$ where $\alpha\colon G/H \to G/K$ is a G-map, $w\colon G/H \times I \to B$ is a G-homotopy from x to $y \circ \alpha$, and $[w]$ is the homotopy class of G-homotopies from x to $y \circ \alpha$ relative $G/H \times \{0,1\}$.*
- *composition given by $(\alpha_2, [w_2]) \circ (\alpha_1, [w_1]) \coloneqq (\alpha_2 \circ \alpha_1, [w_2(\alpha_1 \times \mathrm{id}_I) \star w_1])$, where*

$$w_2(\alpha_1 \times \mathrm{id}_I) \star w_1\colon\ G/H \times I \to B,\ (gH, t) \mapsto \begin{cases} w_1(gH, 2t) & t \in [0, \frac{1}{2}] \\ w_2(\alpha_1(gH), 2t-1) & t \in (\frac{1}{2}, 1] \end{cases}$$

Remark 1.8 (Objects in ΠB). Let $x\colon G/H \to B$ be an object of ΠB. Since x is a G-map we get for all $g \in G$ that $x(gH) = x(g \cdot eH) = g \cdot x(eH)$. So x is completely determined by its value at eH. Further, if $g \in H$ it follows $x(eH) = x(gH) = g \cdot x(eH)$. So $x(eH) \in B^H$, the H-fixed points of B. Therefore such an object x in B can be thought of as a point in B^H together with its orbit.

Remark 1.9 (Morphisms in ΠB). Let $x\colon G/H \to B$ and $y\colon G/K \to B$ be objects in ΠB and $(\alpha, [w])$ a morphism from x to y. Note that for all $t \in [0,1]$ we have that $w(\cdot, t)\colon G/H \to B$ is a G-map. Therefore, for every $g \in G$ we have that $w(gH, \cdot)\colon [0,1] \to B$ is a path in B^H from $x(gH) = gx(eH)$ to $(y \circ \alpha)(gH) = g(y \circ \alpha)(eH)$. Therefore similarly to the objects, such a morphism $(\alpha, [w])$ can be thought of as a homotopy class of paths from $x(eH)$ to $(y(\alpha(eH))$ in B^H together with its images under the group action. The meaning of α here is to choose to which of the points in the image of y these paths go, whereas $[w]$ gives their path homotopy classes in B^H.

Remark 1.10 (Composition of morphisms in ΠB). Let $x\colon G/H \to B$, $y\colon G/K \to B$ and $z\colon G/L \to B$ objects in ΠB, $(\alpha_1, [w_1]) \in \Pi B(x, y)$ and $(\alpha_2, [w_2]) \in \Pi B(y, z)$ morphisms. Let $(\alpha, [w]) \coloneqq (\alpha_2, [w_2]) \circ (\alpha_1, [w_1])$ with w as in the definition of composition. With the intuition on objects and morphisms of ΠB as points and paths together with their images under the group action, the composition of two morphisms should represent the composition of these paths. We focus on the composite path starting at $x(eH)$, as by the previous remark this already determines w. The first path here is $w_1(eH, \cdot)$ from $x(eH)$ to $y(\alpha_1(eH))$. Note that in general $\alpha_1(eH) \neq eK$, so we can not compose the first path with $w_2(eK, \cdot)$ but have to choose $w_2(\alpha_1(eH), \cdot)$ from $y(\alpha_1(eH))$ to $z(\alpha_2(\alpha_1(eH)))$ as second path. Comparing with the definition of composition we find that this is exactly what happens:

$$w(eH,t) = \begin{cases} w_1(eH,2t) & t \in [0,\frac{1}{2}] \\ w_2(\alpha_1(eH),2t-1) & t \in (\frac{1}{2},1] \end{cases}$$

Composition is well defined in the same way as composition of path-homotopy classes are well defined in a (not equivariant) fundamental groupoid.

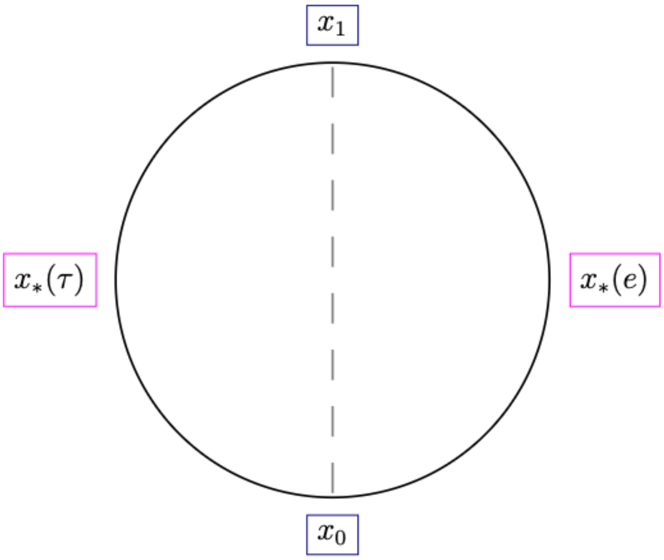
There is no clear way on how to write down a specific equivariant fundamental groupoid, as in general B has infinitely many points and therefore ΠB infinitely many objects. So in calculation we try to determine a skeleton of ΠB instead, which can be thought of as the smallest equivalent category.

Definition 1.11 (Skeleton)**.** *Let C be a category. A subcategory D of C is called* skeleton of C *if it is equivalent to C and any two isomorphic objects of D are equal.*

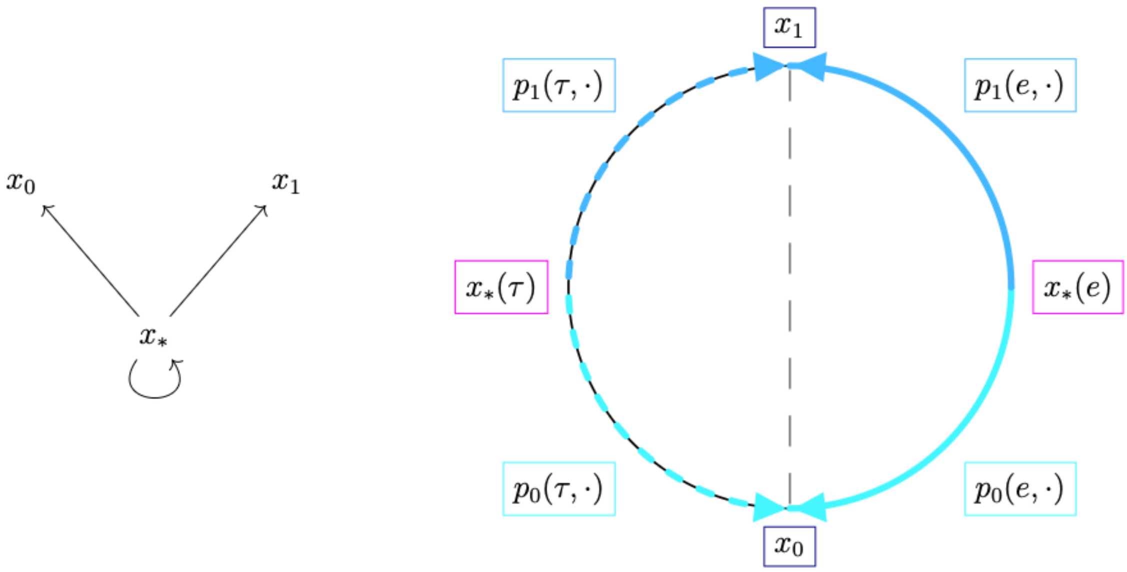
Let's have a look at examples.

Example 1.12 ($\Pi S^{1,1}$ (Example 2.10 in [BLM$^+$24])**.**) Let $S^{1,1}$ denote the C_2-space with underlying space $i_e^* S^{1,1} = S^1$ and involution $\tau\colon S^1 \to S^1, (x,y) \mapsto (-x,y)$, which is mirroring S^1 on the vertical axis. Then let $x_* \in \Pi S^{1,1}$ of the form $C_2/e \to S^1$ with $x_*(e) = (1,0)$ and therefore $x_*(\tau) = (-1,0)$. Then any other $x \in \Pi S^{1,1}$ of the form $C_2/e \to S^1$ is isomorphic to x_*, since S^1 is path connected and we can find a path from $x_*(e)$ to $x(e)$, defining a morphism $(e,[w])$ from x_* to x, which is inverted by the morphism defined by the reverse path. This means that in a skeleton of $\Pi S^{1,1}$ we only have one object of this form, and we choose x_* for that.

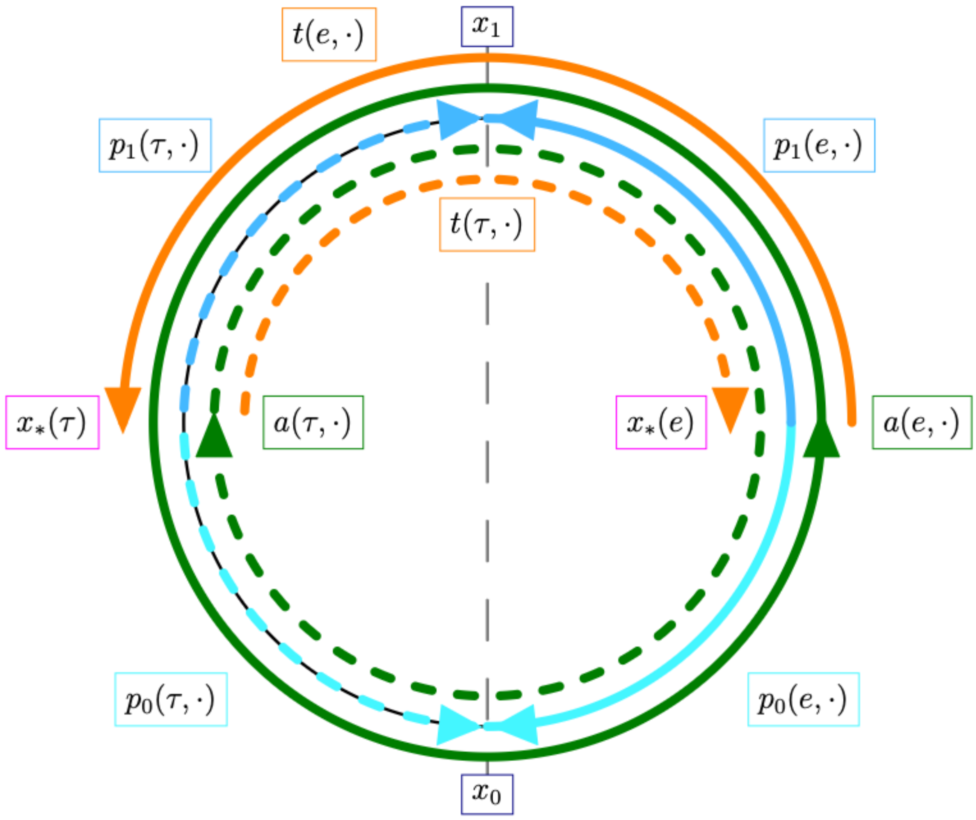
Further, we have that the fixed set of the involution is $(S^{1,1})^{C_2} = \{(0,1),(0,-1)\}$, consisting of two points. For each we get an object of a skeleton of $\Pi S^{1,1}$, namely $x_0\colon C_2/C_2 \to S_1$ with $x_0(eC_2) = (0,-1)$ and $x_1\colon C_2/C_2 \to S_1$ with $x_1(eC_2) = (0,1)$. Because x_0 and x_1 are really just picking one point, we will just write x_i for $x_i(eC_2)$. The two are not isomorphic, as a morphism between the two would require a path in $(S^{1,1})^{C_2} = \{x_0,x_1\}$ between x_0 and x_1.



Since C_2 and $\{e\}$ are the only subgroups of C_2, it follows that all objects in $\Pi S^{1,1}$ are either x_0, x_1 or isomorphic to x_*. Further, since there are no G-maps from C_2/C_2 to C_2/e, there are no morphisms from x_0 or x_1 to x_*. So they together with the morphisms between them form a skeleton of the following form (omitting the id-arrows at the x_i):

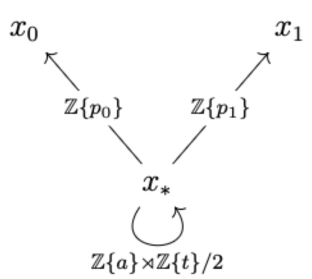


To find the morphisms from x_* to x_i, $i \in \{0, 1\}$, note that the quotient $\rho\colon C_2 \to C_2/C_2$ is the only C_2-map from C_2 to C_2/C_2. Further, we fix the morphisms $(\rho, [p_i])$ such that $p_i(e, \cdot)$ are the paths from $x_*(e)$ to x_i as in the picture. Then each path up to path homotopy from $x_*(e)$ to x_i is given by pre-composing $p_i(e, \cdot)$ with a loop in $\pi_1(S^1, x_*(e))$ and each of those composed paths defines a morphism from x_* to x_i. This turns $\Pi S^{1,1}(x_*, x_i)$ into a right $\pi_1(S^{1,1}, x_*(e))$-module, and we write $\Pi S^{1,1}(x_*, x_i) \approx \mathbb{Z}\{p_i\}$.



To find the automorphisms of x_*, we fix the morphisms $(e,[a])$ and $(\tau,[t])$ defined by the loop $a(e,\cdot)$ at $x_*(e)$ which generates $\pi_1(S^1,x_*(e))$ and the path $t(e,\cdot)$ from $x_*(e)$ to $x_*(\tau)$ as in the picture. Then by looking at the picture we find the relations $(\tau,[t])\circ(\tau,[t]) = \mathrm{id}$ and $(\tau,[t])\circ(e,[a])\circ(\tau,[t]) = (e,[a])^{-1}$. We abuse notation and write $t^2 = \mathrm{id}$ and $tat = a^{-1}$ for short. Note that it suffices to check the relations for the path starting at $x_*(e)$, the one starting at $x_*(\tau)$ is fully determined and follows the same relations. Formally one can calculate e.g. $((\tau,[t])\circ(\tau,[t])) = (\tau\tau,[t(\tau\times\mathrm{id})\star t]) = (e,[t^{-1}\star t]) = (e,[c_{x_*}])$ where t^{-1} and c_{x_*} are defined by $t^{-1}(e,\cdot)$ being the reverse path from $t(e,\cdot)$ and $c_{x_*}(e,\cdot)$ being the constant path at $x_*(e)$. So we get that $\Pi S^{1,1}(x_*,x_*) \cong < t,a|t^2=\mathrm{id}, tat=a^{-1} > \cong \mathbb{Z}\{a\}\rtimes\mathbb{Z}\{t\}/2$ is the non-trivial semidirect product of $\mathbb{Z}/2$ and $\mathbb{Z}$.

Putting everything together we get the following skeleton of $\Pi S^{1,1}$:

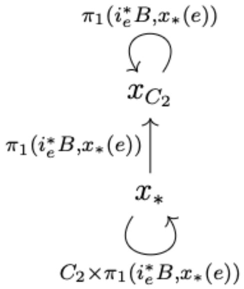


Example 1.13 (Trivial path connected G-space (Example 2.12 in [BLM+24])). Let B be a path-connected G-space with trivial G-action. Then for every subgroup H of G the H-fixed points are the whole space $B^H = i_e^* B$ and especially path-connected. So as in the previous example we get that a skeleton of ΠB has one object for each quotient G/H. They could be described as follows: Fix any object $x_*: G/G \to B$, and for each subgroup H define the composition $x_H: G/H \xrightarrow{\rho} G/G \xrightarrow{x_*} B$, where ρ is the quotient.

For each choice of G-map $\alpha: G/H \to G/K$ and each choice of $[p] \in \pi_1(i_e^* B, x_*(e))$ one gets a unique morphism $(\alpha, [w_p]) \in \Pi B(x_H, x_K)$ where $w_p(eH, \cdot) = p$. These are already all morphisms in $\Pi B(x_H, x_K)$, so if $\mathcal{O}(G/H, G/K)$ denotes the set of G-maps from G/H to G/K, then $\Pi B(x_H, x_K)$ is in bijection to $\mathcal{O}(G/H, G/K) \times \pi_1(i_e^* B, x_*(e))$.

Especially for G/e there is a group isomorphism $\Pi B(x_e, x_e) \cong G \times \pi_1(i_e^* B, x_*(e))$.

So for $G = C_2$ one obtains the following skeleton of ΠB:



1.2 Closed surfaces

By a *closed surface* we mean a compact surface without boundary. There is a classical result stating that all connected closed surfaces are either homeomorphic to the sphere, a connected sum of tori or a connected sum of projective planes. This result relies on the fact, that every closed surface can be triangulated. Its history and various proofs can be looked up in [GX13]. In this section we will give the classification without proof and repeat some known facts and intuitions about the closed surfaces which we will need later.

Definition 1.14 (connected sum). *Let M_1, M_2 be connected closed surfaces, D^2 the closed 2-disk with boundary S^1 and let $h_i: D^2 \to M_i$, $i = 1, 2$ be two embeddings. Then the* connected sum of M_1 and M_2 *is denoted by $M_1 \# M_2$ and given by removing the interiors of the images of the disk and gluing along their boundaries:*

$$M_1 \# M_2 := (M_1 - h_1(\text{int} D^2)) \cup_f (M_2 - h_2(\text{int} D^2))$$

where $f\colon h_1(S^1) \to h_2(S^1)$ *is the homeomorphism* $h_2 h_1^{-1}$ *restricted to* S_1*.*

Remark 1.15. This construction does not depend on the choice of h_1 and h_2 in the sense, that all resulting connected sums are homeomorphic.

Example 1.16. Let's look at $T_2 \coloneqq T\#T$, the connected sum of two tori. The 3-dimensional visualization can look something like in the pictures, creating something like a figure 8. The right picture also shows a nice set of generators of $\pi_1(T_2)$.

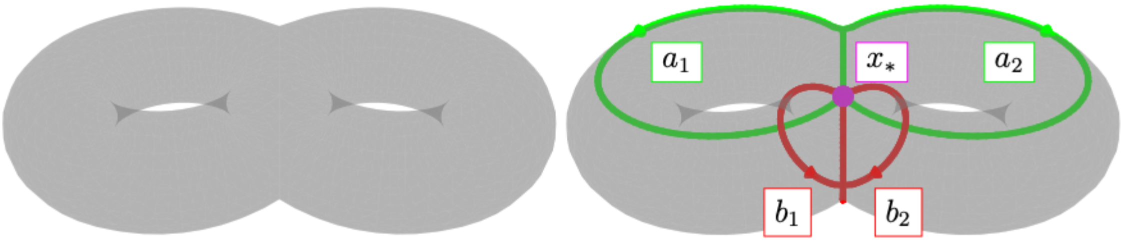


Figure 8

Remark 1.17. Because every closed surface can be triangulated, one can turn it into a homeomorphic polygon with pairwise identified edges:

Let $\Delta_1, \ldots, \Delta_n$ be the triangles in a triangulation of a closed surface M. W.l.o.g the ordering is such that for all $i \in \{2, \ldots, n\}$ $\cup_{j<i}\Delta_j$ and Δ_i share at least one edge E_i. Define $F_n = \Delta_1 \cup_{E_2} \Delta_2 \cup_{E_3} \cdots \cup_{E_n} \Delta_n$. Then F_n is homeomorphic to the disk D^2 and can be thought of as a $(n+2)$-gon. M is obtained from F_n by identifying its edges pairwise in the way given by the triangulation.

Example 1.18. Let's look at $T_2 = T\#T$ again, but in the setting of the tori viewed as squares with identified edges. A convenient way to think about the process of taking the connected sum is the following: In both squares, cut out the inside of a loop starting at one of the corners of the square. Then again on both squares, split the start- and endpoint of the loop (this is okay as they will still be identified by the edge identifications) and straighten out the loop, creating a polygon with 5 edges. Now glue the 5-gons together at the border of the loop, creating the typical 8-gon representing T_2. See Figure 1.1

Example 1.19. The projective plane N can be understood as a 2-gon with the identified arrows forming a loop. Then $N_2 \coloneqq N\#N$ is represented by a 4-gon which turns out to be the Klein bottle.

Remark 1.20. This construction works for any two closed surfaces. Especially we get that the connected sum of g many tori T_g is represented by a $4g$-gon constructed by iteratively taking the connected sum, and the connected sum of g many projective planes N_g is represented by a $2g$-gon.

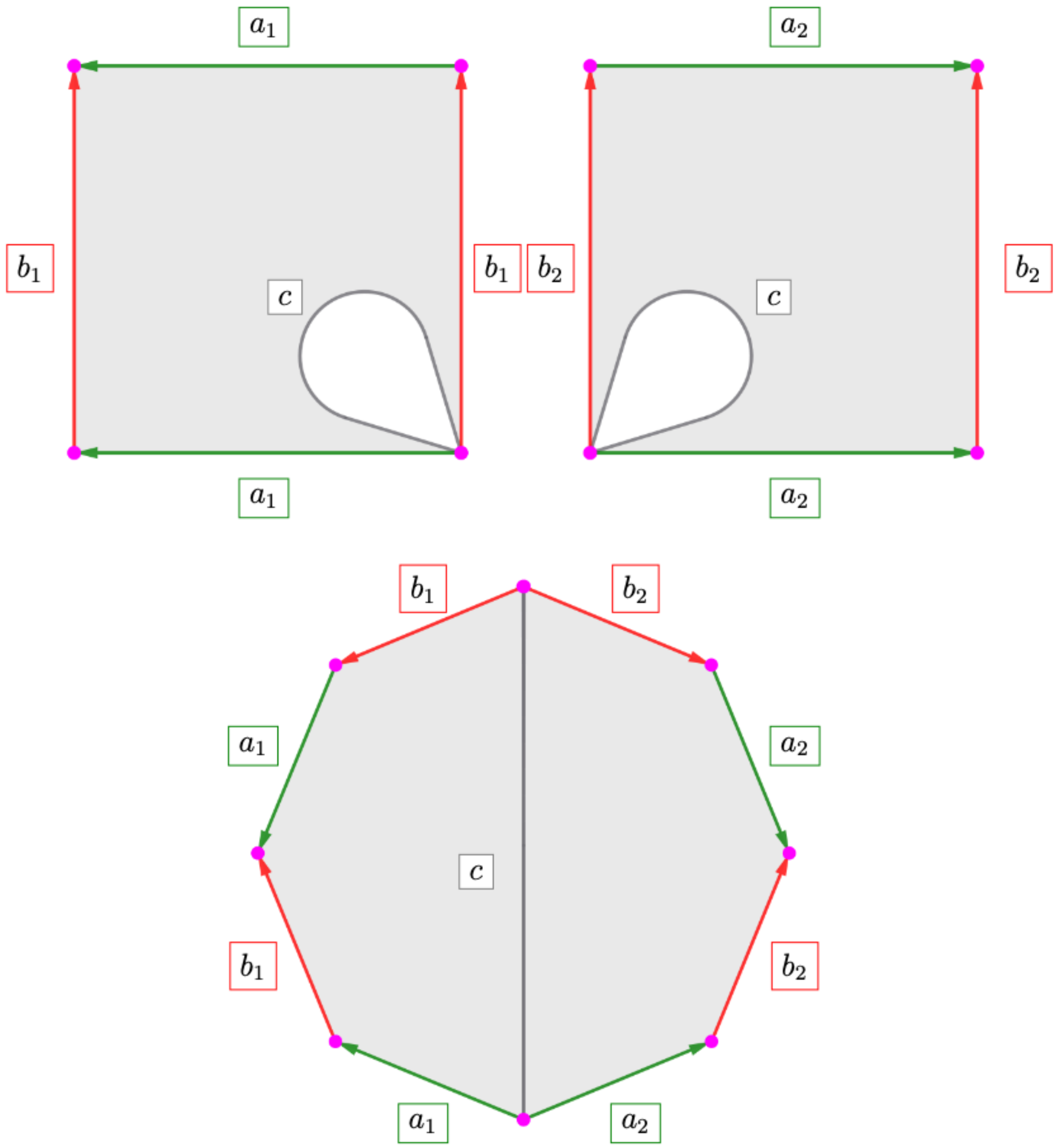


Figure 1.1: $T^{\text{refl}}_{2,1}$-gon

Note that for the order and direction of the identified arrows it does make a difference at which of the corners of the two polygons one cuts out the loop. So there are a bunch of different $4g$-gons all representing T_g. For calculating the equivariant fundamental groupoid it is important to choose a representation which fits nicely with the group action. E.g. the representation from Example 1.18 fits with the choice of generators in Example 1.16 and is well suited for calculating the fundamental groupoid $\Pi T_{2,1}^{\text{refl}}$, where $T_{2,1}^{\text{refl}}$ is T_2 with reflection along the glued circle c.

Remark 1.21. Working with the polygon representation of closed surfaces, one can find different homeomorphisms between polygons with different boundary words. This is one of the main ideas for proving the classification of closed surfaces.

Theorem 1.22 (Classification of closed surfaces)**.** *Every connected closed surface is homeomorphic to one of the following:*

- *the sphere, denoted by* S^2
- *a connected sum of* $g \geq 1$ *many tori, denoted by* T_g
- *a connected sum of* $g \geq 1$ *many real projective planes, denoted by* N_g.

We will need the fundamental groups of all closed surfaces. These can be calculated with the Seifert-Van Kampen theorem, which we will revise here.

Definition 1.23 (presentation of a group)**.** *Let* S *be a set and let* $F(S)$ *denote the free group on* S*. Further, let* R *be a subset of* $F(S)$*, i.e. a set of words on* S*, and* N *the smallest normal subgroup of* $F(S)$ *with* $R \subset N$*. Then we define* $< S|R > := F(S)/N$*. We say that a group* G *has presentation* $< S|R >$ *if* $G \cong < S|R >$*.*

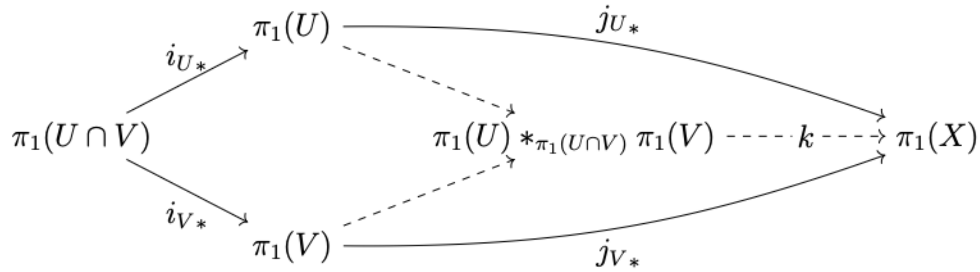
Remark 1.24. Every group has a presentation. Often we will write relations in the form $w_1 = w_2$ where w_1 and w_2 are words on S, meaning $w_2^{-1} w_1 \in R$.

Theorem 1.25 (free product with amalgamation)**.** *Let* A, G, H *be groups and* $\phi : A \to G$*,* $\psi : A \to H$ *group homomorphisms. Further, choose presentations* $A = < S_A|R_A >$*,* $G = < S_G|R_G >$ *and* $H = < S_H|R_H >$*.*

Then the pushout of (ϕ, ψ) *in the category of groups, denoted by* $G *_A H$ *and called free product with amalgamation, is given by* $G *_A H = < S_G \cup S_H | R_G \cup R_H \cup R >$ *where* R *contains the relations* $\phi(a) = \psi(a)$ *for all* $a \in S_A$*.*

Now we can state the Seifert-Van Kampen theorem:

Theorem 1.26 (Seifert-Van Kampen [Kam])**.** *Let* X *be a topological space and* $U, V \subseteq X$ *open and path-connected with* $X = U \cup V$ *and* $U \cap V$ *path-connected and non-empty. Choose the base point of fundamental groups as* $x_0 \in U \cap V$ *and let* $i_U : U \cap V \to U$*,* $i_V : V \cap U \to V$*,* $j_U : U \to X$*,* $j_V \to Y$ *be inclusions. Then* k *in* *the following pushout diagram is an isomorphism:*



Now we can calculate the fundamental groups of closed surfaces.

Theorem 1.27. *Let M be a closed surface. Let M be represented by a $2n$-gon with boundary word W in the edge names $(a_i)_{i\leq n}$. Then the fundamental group of M is $\pi_1(M) =< (a_i)_{i\leq n} | W = \mathrm{id} >$.*

Proof. Cover M by an open disk U in the interior of the polygon and $V = M - \{u\}$ where $u \in U$. Then U is contractible and $\pi_1(U) = \{e\}$. $U \cap V = U \setminus \{u\}$ is as punctured disk homotopy equivalent to S^1 and therefore $\pi_1(U \cap V) \cong \mathbb{Z}$. V is homotopy equivalent to the boundary of the $2n$-gon. After the identifications the boundary is homeomorphic to $\bigvee_n S^1$, the wedge sum of n many S^1. So $\pi_1(V) \cong< (a_i)_{i\leq n}| >\cong *_n\mathbb{Z}$

The group homomorphism induced by the inclusion $U \cap V \to U$ is trivial while the group homomorphism induced by the inclusion $U \cap V \to V$ is given by $\mathbb{Z} \cong< g| >\to *_n\mathbb{Z} =< (a_i)_{i\leq n}| >, \quad g \mapsto W$. Intuitively the generator g is walking around u once, it becomes walking around the boundary once.

So by Seifert-van-Kampen we get $\pi_1(M) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \cong< (a_i)_{i\leq n}|W = \mathrm{id} >$. □

Corollary 1.28. *The fundamental group of T_g is*

$$\pi_1(T_g) \cong< (a_i)_{1\leq i\leq g}, (b_i)_{1\leq i\leq g} | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} >$$

and the fundamental group of N_g is

$$\pi_1(N_g) \cong< (a_i)_{1\leq i\leq g} | a_1 a_1 \cdots a_n a_n >$$

1.3 Classification of C_2-surfaces

Definition 1.29 (C_2-surface)**.** *By a* C_2-surface *we mean a C_2-space B, such that the the underlying topological space $i_e^* B$ is a connected closed surface.*

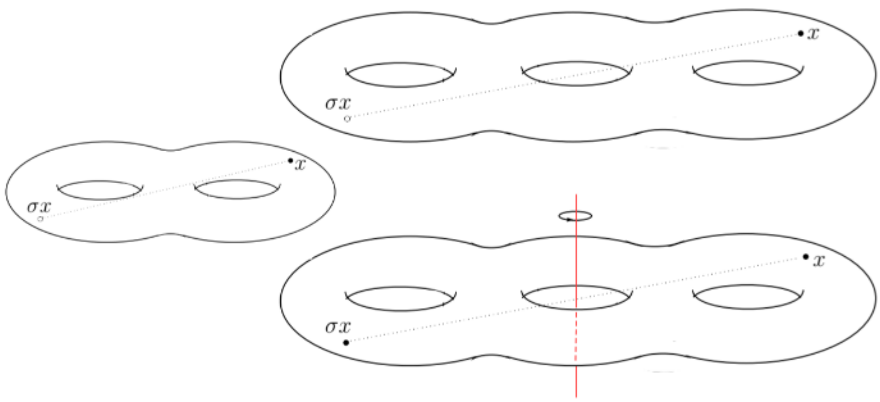


Figure 1.2: Free involutions on T_2 and T_3 [Dug16, 2.7]

C_2-surfaces are fully classified in [Dug16]. We won't repeat the proof of the classification here as this would exceed the scope of the thesis, but rather introduce the necessary concepts and notations to understand the classification of C_2-actions on orientable surfaces. The classification of C_2-actions on non-orientable surfaces is more involved and we won't state it here. This whole chapter is just copying and adjusting the relevant parts from [Dug16].

Let's first construct the free actions of C_2 on T_g as in [Dug16, 2.7]. Let T_g be embedded in $\mathbb{R}^3$ as a straight chain of connected tori as in Figure 1.2, with the "center" of the torus at the origin. Then the antipodal map $x \to -x$ preserves the torus and is an involution. When g is odd the origin is inside the central hole of T_g, and rotation by 180 degrees through the z-axis gives another involution. For T_2 and T_3 these are visualized in Figure 1.2.

These C_2-surfaces are denoted by T_g^{anti} and T_g^{rot} respectively, and are the only free involutions on T_g [Dug16, Theorem 4.1].

Now for involutions with fixed points, consider the constructions of the *spit* action, the *reflection* action and the *anti* action [Dug16, 5.1]:

For $0 \leq r \leq \frac{g}{2}$ let $T_{g,r}^{\text{spit}}$ be the C_2-space as indicated in Figure 1.3. The action is 180 degree rotation around the dotted axis. Note that there is an isolated fixed point at each intersection of T_g with the axis of rotation. Further, the axis passes through $g - 2r$ doughnut holes. This gives a total of $F = 2 + 2g - 4r$ fixed points. Often it is useful to remember the number of fixed points in the notation, so we also write $T_g^{\text{spit}}[F] := T_{g,r(F)}^{\text{spit}}$ where $r(F) := \frac{2+2g-F}{4}$ for the spit action on T_g with F isolated fixed points.

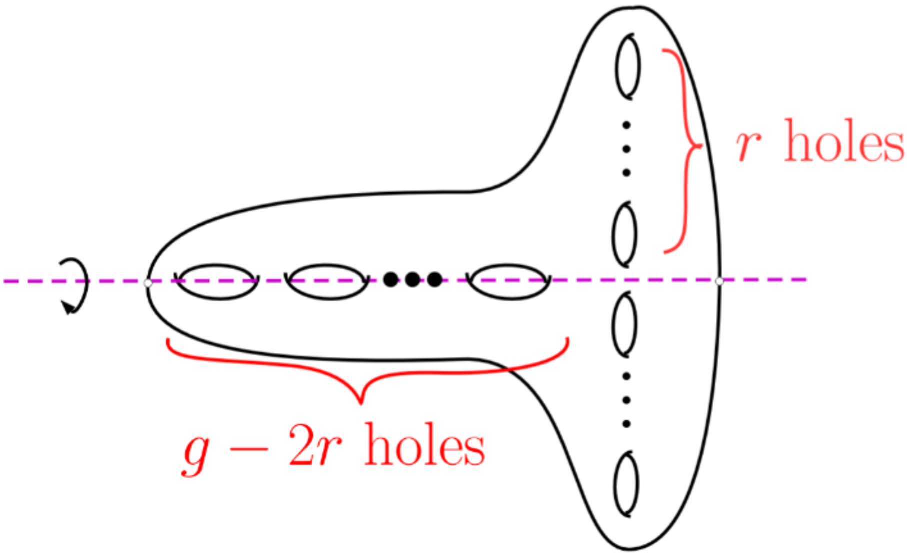


Figure 1.3: *the spit*, $T_{g,r}^{\text{spit}}$, [Dug16, 5.1]

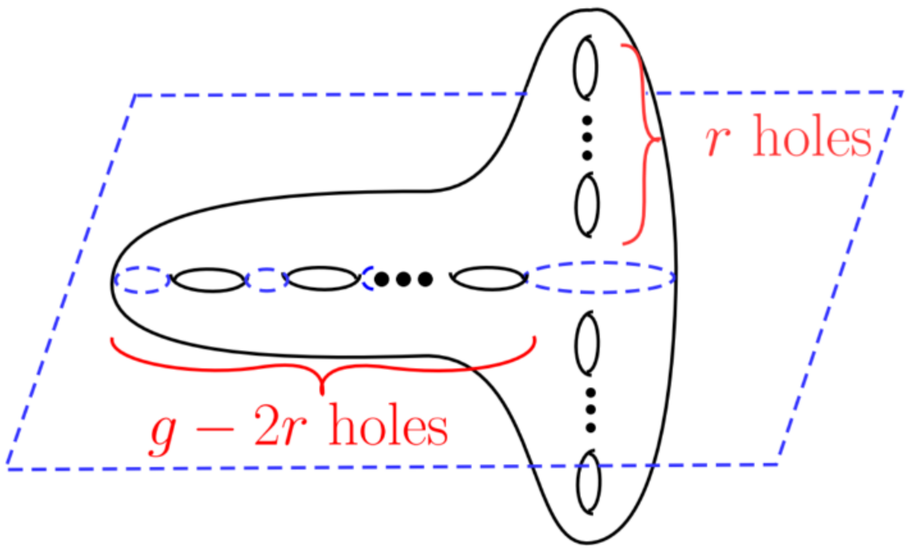
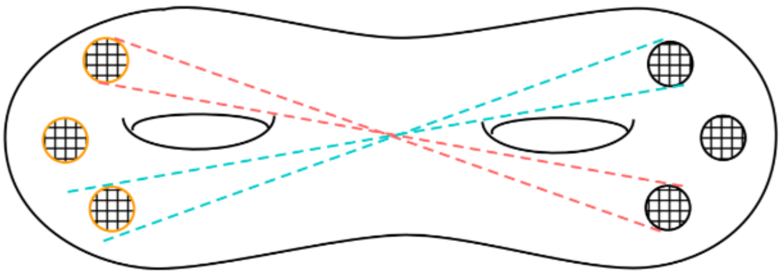


Figure 1.4: *the reflection*, $T_{g,r}^{\text{refl}}$, [Dug16, 5.1]



$T^{\text{anti}}[2,3]$

Figure 1.5: *antipodal action*, $T^{\text{anti}}[2,3]$, [Dug16, 5.1]

Similarly for $0 \leq r \leq \frac{g}{2}$ let $T^{\text{refl}}_{g,r}$ be the C_2-space as indicated in Figure 1.4. The action here is reflection along the xy plane indicated in blue. Note that there are fixed circles wherever T_g intersects with the plane of reflection. The plane passes through $g-2r$ holes, so there are $C = g-2r+1$ fixed circles. Again, sometimes we write $T^{\text{refl}}_g[C] \coloneqq r(C)$ where $r(C) = \frac{1+g-C}{2}$ for the reflection on T_g with C many fixed circles.

Now let $T^{\text{anti}}[g,r]$ denote the C_2-space which can be constructed as follows: Start with T^{anti}_g, i.e. T_g with the antipodal action. Then cut out r many disjoint open disks which are also disjoint with all their images under the involution, and cut out the image disks as well. Finally, identify all points on the boundary of a cut disk with their images under the antipodal action (see Figure 1.5).

This creates r many fixed circles, and the underlying space is $i_e^*(T^{\text{anti}}[g,r]) = T_{g+r}$. One can see that the T^{anti} spaces are different from the T^{refl} spaces by realizing that for the T^{anti} spaces the fixed set doesn't separate the space, while for the T^{refl} spaces it does.

Now we can state the classification of C_2-actions on orientable surfaces.

Theorem 1.30 (Classification of C_2-actions on orientable surfaces (Theorem 5.7 in [Dug16]))**.** *For $g \geq 0$, up to equivariant homeomorphism there are exactly $4g+2$ involutions on T_g (here $T_0 = S^2$). These give rise to the following C_2-surfaces:*

(i) The $2 + \lceil \frac{g}{2} \rceil$ orientation preserving actions, namely

- *The trivial action*
- $T^{\text{spit}}_{g,r}$ *for $0 \leq r \leq \frac{g}{2}$, or equivalently, $T^{\text{spit}}_g[F]$ for $2 \leq F \leq 2+2g$ and $F \equiv 2+2g$ mod 4*
- T^{rot}_g *when g is odd*

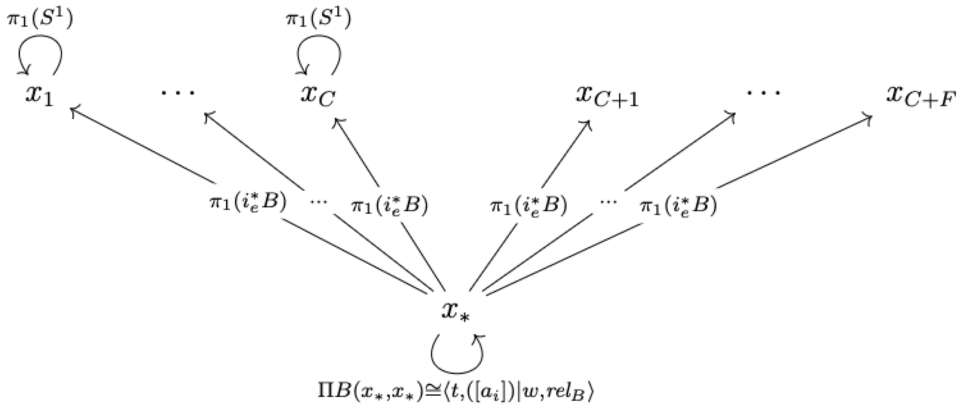
(ii) The $2 + g + \lfloor \frac{g}{2} \rfloor$ orientation-reversing actions, namely

- $T^{\text{refl}}_{g,r}$ *for* $0 \leq r \leq \frac{g}{2}$*, or equivalently* $T^{\text{refl}}_{g}[C]$ *for* $1 \leq C \leq g+1$ *and* $C \equiv g+1$ *mod* 2
- $T^{\text{anti}}[u, g-u]$ *for* $u \leq 0 \leq g$

1.4 The Equivariant fundamental groupoid of C_2-surfaces

Theorem 1.31. *Let B be a C_2-surface with non-trivial action. Further, fix an object x_* in ΠB of the form $C_2/e \to B$ and a path t from $x_*(e)$ to $x_*(\tau)$. Let $\langle (a_i)_{i\in\{1,\dots,n\}} | W \rangle$ be a finite presentation of $\pi_1(i_e^* B)$ as e.g. obtained in 1.27.*

Then for some set rel_B of relations it holds $\Pi B(x_*, x_*) \cong \langle t, (a_i)_{i\in\{1,\dots,n\}} | W, rel_B \rangle$, and ΠB has a skeleton of the following form, where C is the number of fixed circles and F is the number of fixed points:



Further, there is a short exact sequence

$$1 \longrightarrow \pi_1(i_e^* B, x_*(e)) \xrightarrow{i} \Pi B(x_*, x_*) \xrightarrow{\rho} C_2 \longrightarrow 1$$

where $i\colon \pi_1(i_e^* B, x_*(e)) \to \Pi B(x_*, x_*)$ with $\phi([a]) = (e, [v_a])$ where $v_a(e, \cdot) = a$, and ρ is the quotient $\rho\colon \Pi B(x_*, x_*) \to \Pi B(x_*, x_*)/i(\pi_1(i_e^* B)) \cong C_2$ sending $(\alpha, [w]) \mapsto \alpha$.

Proof. Let B be a C_2-surface with non-trivial action. Since we assume connectedness, all objects in ΠB of the form $C_2/e \to B$ are isomorphic, and therefore a skeleton of ΠB contains only one of them. We will always call this one x_*. Further, the skeleton has one object of the form $C_2/C_2 \to B$ for each connected component of $(B)^{C_2}$, the fixed points of B. We will call these objects x_i.

Depending on if the connected component is a fixed point or a fixed circle it has either only the trivial automorphism or its automorphism group is isomorphic to $\pi_1(S^1) \cong \mathbb{Z}$.

Further, there exist morphisms from x_* to the x_i. Each set of morphisms $\Pi B(x_*, x_i)$ is in bijection to $\pi_1(i_e^* B)$, which can be seen as follows:

Every element $(\rho, [w]) \in \Pi B(x_*, x_i)$ corresponds one to one to a path homotopy class of paths from $x_*(e)$ to x_i, as the quotient $\rho\colon C_2/e \to C_2/C_2$ is the only C_2-map from C_2/e to C_2/C_2 and because $[w]$ is fully determined by the path homotopy class of $w(e, \cdot)$. And for any such path-homotopy class one gets a morphism $(\rho, [w])$ by choosing $w(e, \cdot)$ to be in it. Since B is path-connected, we know that the set of path homotopy classes of paths between two given points is in bijection to the fundamental group $\pi_1(i_e^* B, x_*)$.

The only remaining and interesting morphisms in a skeleton of ΠB therefore are those in $\Pi B(x_*, x_*)$, i.e. automorphisms of x_*.

There is an injective group homomorphism $i\colon \pi_1(i_e^* B, x_*(e)) \to \Pi B(x_*, x_*)$ with $\phi([a]) = (e, [v_a])$ where $v_a(e, \cdot) = a$. This is a group isomorphism to the subgroup $\{(\alpha, [v]) \in \Pi B(x_*, x_*) : \alpha = e\}$, which in turn is the kernel of $\rho\colon \Pi B(x_*, x_*) \to C_2, \quad (\alpha, v) \mapsto \alpha$. So we get the short exact sequence as in the theorem. Further, we get that if $\pi_1(i_e^* B, x_*(e))$ is generated by $([a_i])_{i \in \{1,\dots,n\}}$ and we choose any $(\tau, [t]) \in \Pi B(x_*, x_*)$, then $\Pi B(x_*, x_*)$ is generated by $\{(\tau, [t]), (\phi([a_i]))_{i \in \{1,\dots,n\}}\}$, which yields the presentation of $\Pi B(x_*, x_*)$ as in the theorem.

Putting everything together we get the diagram above. □

Remark 1.32. Note that if in the situation of the above theorem we have that rel_B contains the relation $t^2 = \mathrm{id}$, i.e. that $(\tau, [t])^2 = \mathrm{id}$, the map $s\colon C_2 \to \Pi B(x_*, x_*), \quad \tau \mapsto (\tau, [t])$ is a group homomorphism with $\rho \circ s = \mathrm{id}_{C_2}$. By a version of the splitting lemma this yields that $\Pi B(x_*, x_*)$ is a semidirect product $\pi_1(i_e^* B, x_*(e)) \rtimes_\varphi C_2$ where $\varphi\colon C_2 \to \mathrm{Aut}(\pi_1(i_e^* B, x_*(e)))$ is given by $\varphi(\tau)([a]) = i^{-1}((\tau, [t]) i([a]) (\tau, [t]))$.

If on the other hand we he have that $\Pi B(x_*, x_*) = \pi_1(i_e^* B, x_*(e)) \rtimes_\varphi C_2$ for some φ, then it follows by the splitting lemma that there exists a group homomorphism $s : C_2 \to \Pi B(x_*, x_*)$ with $\rho \circ s = \mathrm{id}$. This implies that $s(\tau)^2 = s(\tau^2) = \mathrm{id}$, i.e. we get the existence of the morphism $(\tau, [t]) \coloneqq s(\tau)$ with $(\tau, [t])^2 = \mathrm{id}$.

Theorem 1.33. *Let B be a C_2-surface. Then $\Pi B(x_*, x_*)$ is a semidirect product $\pi_1(i_e^* B, x_*(e)) \rtimes_\varphi C_2$ if and only if B has fixed points or $B = S^2_a$.*

Proof. In case the action on B is trivial, B has fixed points and $\Pi B(x_*, x_*) \cong C_2 \times \pi_1(i_e^* B, x_*(e))$ by 1.13.

In case the action on B is non-trivial, the theorem and the remark above hold. By the remark we get that $\Pi B(x_*, x_*)$ is a semidirect product $\pi_1(i_e^* B, x_*(e)) \rtimes_\varphi C_2$ if and only if we have $(\tau, [t])^2 = \mathrm{id}$ for some morphism $(\tau, [t]) \in \Pi B(x_*, x_*)$.

First assume that B has a fixed point x_0. Then x_* and t can be chosen such that $x_*(e) = x_*(\tau) = x_0$ and that $t = c_{x_0}$ is the constant path at x_0. Then $(\tau, [t])^2 = (\tau, [c_{x_0}])^2 = (e, [c_{x_0}]) = \mathrm{id} \in \Pi B(x_*, x_*)$, so by the remark we get that $\Pi B(x_*, x_*)$ is a semidirect product $\pi_1(i_e^* B, x_*(e)) \rtimes_\varphi C_2$.

For the case $B = S_a^2$ Theorem 2.1 gives us $\Pi S_a^2(x_*, x_*) \cong C_2 \cong C_2 \times \pi_1(S^2)$.

For the other direction assume that $\Pi B(x_*, x_*)$ is a semidirect product $\pi_1(i_e^* B, x_*(e)) \rtimes_\varphi C_2$. Then there exists a morphism with $(\tau, [t])^2 = \mathrm{id} = (e, [c_{x_*}])$ where $c_{x_*}(e, \cdot)$ is the constant path at $x_*(e)$.

Note that by $I/_{\{0,1\}} \cong S_1$ we get that $t(\tau \times \mathrm{id}) \star t \colon C_2 \times I \to B$ factors through $C_2 \times S^1 \to B$. Further, let $i \colon C_2 \times S^1 \to C_2 \times D^2$ be the inclusion of the S^1's to the boundaries of the respective disks. Then, since $t(\tau \times \mathrm{id}) \star t$ is equivariant path-homotopic to the constant paths c_{x_*}, we know that it extends to an equivariant map $h \colon C_2 \times D^2 \to B$ such that $h \circ i = t(\tau \times \mathrm{id}) \star t$:

$$\begin{array}{ccc} C_2 \times S_1 & \xrightarrow{t(\tau \times \mathrm{id}) \star t} & B \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle h} & \\ C_2 \times D^2 & & \end{array}$$

$(t(\tau \times \mathrm{id}) \star t)(e, \cdot)$ is the path $t(e, \cdot)$ concatenated with its image $t(\tau, \cdot)$. The same is true with e and τ flipped. So by choosing a uniform identification of $I/_{\{0,1\}} \cong S_1$ we get for all $x \in S^1$ that $(t(\tau \times \mathrm{id}) \star t)(e, x) = (t(\tau \times \mathrm{id}) \star t)(\tau, -x)$. Identifying $(e, x) \sim (\tau, -x)$, we get that $t(\tau \times \mathrm{id}) \star t \colon C_2 \times S^1 \to B$ factors through $\tilde{t} : S_a^1 \to B$, where S_a^1 is S^1 with the antipodal action.

Therefore we can identify the boundaries of the two disks in $C_2 \times D^2$ such that the boundary becomes S_a^1, obtaining a space isomorphic to S_a^2 via

$$f \colon \; C_2 \times D^2/_{(e,x) \sim (\tau, -x) \;\; \forall x \in \partial D} \to S_a^2 \quad (e, (x_1, x_2)) \mapsto (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$$

where $\{e\} \times D^2$ becomes the northern hemisphere and $\{\tau\} \times (-D^2)$ becomes the southern hemisphere. Then h factors through $\tilde{h} \colon S_a^2 \to B$. Let $\tilde{i} \colon S_a^1 \to S_a^2$ be the inclusion at the equator, then:

$$\begin{array}{ccc} S^1_a & \xrightarrow{\tilde{i}} & B \\ {\scriptstyle \tilde{i}}\downarrow & \overset{\tilde{h}}{\nearrow} & \\ S^2_a & & \end{array}$$

Further, let $p\colon E \to i_e^*B$ be a universal cover of i_e^*B. For now let's treat the case $i_e^*B \notin \{S^2, N\}$, i.e. the underlying space is neither the sphere S^2 nor the projective plane N. Then either $i_e^*B \in \{T, N_2\}$ is the torus T or the Klein bottle N_2, in which case $E \cong \mathbb{R}^2$ is a universal cover of i_e^*B, or $E \cong \mathbb{H}^2(\cong \mathbb{R}^2)$ the hyperbolic plane is the universal cover (see Remark 1.34 below).

Since $\pi_1(S^2) = \{e\}$, we get a lift $(i_e^*\tilde{h})'\colon S^2 \to E$ of $i_e^*\tilde{h}$ by the lifting lemma:

$$\begin{array}{ccc} & & E \\ & \overset{(i_e^*\tilde{h})'}{\nearrow} & \downarrow{\scriptstyle p} \\ S^2 & \xrightarrow{i_e^*\tilde{h}} & B \end{array}$$

Then by the Borsuk–Ulam theorem there exists an $x \in S^2$ such that $(i_e^*\tilde{h})'(x) = (i_e^*\tilde{h})'(-x)$. It follows $i_e^*\tilde{h}(x) = i_e^*\tilde{h}(-x)$. This means, that $i_e^*\tilde{h}(x)$ is a fixed point of B: $\tau(i_e^*\tilde{h}(x)) = i_e^*\tilde{h}(\tau(x)) = i_e^*\tilde{h}(-x) = i_e^*\tilde{h}(x)$.

For the case $i_e^*B = S^2$ we check that the only C_2-sphere without fixed points is S^2_a (see Theorem 2.1).

For the case $i_e^*B = N$ is the projective plane, there are just two possibilities for B [Dug16], namely N with the trivial action and the space obtained from $S^{2,1}$ or $S^{2,2}$ (see Section 2.1) by identifying antipodal points. Both have fixed points. □

Remark 1.34. The result, that for $g \geq 1$ $\mathbb{R}^2$ is a universal covering of T_g and N_{g+1}, follows from the Cartan–Hadamard Theorem [Car] in Riemannian geometry, which uses the non-positive curvature of these manifolds. A more explicit way to see how these coverings work is to notice that in the same way $\mathbb{R}^2$ can be tiled by squares, $\mathbb{H}^2$ can be tiled by regular $4g$-gons such that $4g$-many edges meet at a corner for $g \geq 2$ (Tiling with Schläfi-symbol $\{4g, 4g\}$). So T_g is a suitable quotient of $\mathbb{H}^2$ and $\mathbb{H}^2$ their universal cover [UCS]. Since T_g is a two-sheeted covering of N_{g+1}, we also get that $\mathbb{H}^2$ is a universal cover of N_{g+1} for $g \geq 2$.

Remark 1.35. The essential and potentially difficult part of computing the equivariant fundamental groupoid ΠB of a C_2-surface is to make suitable choices for x_* and the $[a_i]$ generating $\pi_1(i_e^*B, x_*(e))$, such that the relations in rel_B take a nice form and can be found easily. If B has fixed points, choosing x_* as one of those is usually a good choice.

The group $\Pi B(x_*, x_*)$ can be understood to capture how the involution acts on the fundamental group of $i_e^* B$.

There remain the following open questions:

Question 1.36. *Is a C_2-surface uniquely determined by its equivariant fundamental groupoid and the fundamental groupoid of its underlying surface?*

Question 1.37. *Let X be a closed surface. Do all possible semidirect products of the form $\pi_1(X) \rtimes_\varphi C_2$ occur as $\Pi B(x_*, x_*)$ of some C_2-surface B with $i_e^* B = X$?*

We will answer these two questions in the cases $i_e^* B = X = S^2$ and $i_e^* B = X = T$ in Chapter 2.

Chapter 2

Calculations

2.1 Sphere

Classification of C_2-actions on orientable surfaces (see Theorem [1.30]) there are four involutions on $T_0 = S^2$. These are the trivial action, the reflection along a plane, the half rotation about an axis, and the antipodal action. The corresponding C_2-surfaces (C_2-spheres) are denoted by $S^{2,0} \coloneqq T_0^{\text{trivial}}$, $S^{2,1} \coloneqq T_0^{\text{refl}}[C = 1]$, $S^{2,2} \coloneqq T_0^{\text{spit}}[F = 2]$ and $S_a^2 \coloneqq T_0^{\text{anti}}$. They can be thought of as reflections of S^2 at $0, 1, 2$ or 3 of the coordinate system planes.

Let B denote any C_2-sphere. Recall the short exact sequence from Theorem 1.31:

$$1 \longrightarrow \pi_1(i_e^* B, x_*(e)) \xrightarrow{\ i\ } \Pi B(x_*, x_*) \xrightarrow{\ \rho\ } C_2 \longrightarrow 1$$

Since $i_e^* B = S^2$ is simply connected, the exact sequence simplifies to

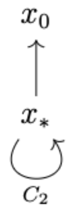
$$1 \longrightarrow \Pi B(x_*, x_*) \longrightarrow C_2 \longrightarrow 1$$

which yields $\Pi B(x_*, x_*) \cong C_2$.

So together with the information about the fixed set of these C_2-surfaces, Theorem 1.30 and Example 1.13, we get the following skeletons of equivariant fundamental groupoids of C_2-spheres:

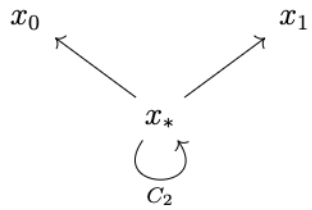
Theorem 2.1 (The equivariant fundamental groupoids of C_2-spheres)**.** *The C_2-spheres have equivariant fundamental groupoids with the following skeletons, i.p. the equivariant fundamental groupoid distinguishes them:*

$\Pi S^{2,0}$ *(trivial aciton):*



$\Pi S^{2,1}$ *(reflection):*

$\Pi S^{2,2}$ *(rotation):*



ΠS^2_a *(antipodal action):*

2.2 Torus

There are 6 C_2-tori by Theorem 1.30. These are:

- T_1^{trivial}, the trivial action
- $T_1^{\text{spit}}[F = 4]$, the rotation with 4 fixed points
- T_1^{rot}, the free rotation
- $T_1^{\text{refl}}[C = 2]$, the reflection
- $T^{\text{anti}}[1, 0]$, the antipodal action
- $T^{\text{anti}}[0, 1]$, the toilet paper roll

We will calculate their equivariant fundamental groupoids with the help of Theorem 1.31 and switching back and forth between the embedded representation of the torus and the representation as square with identified edges. In the case of actions with fixed points we also want to look at the possible representation of $\Pi B(x_*, x_*)$ as a semidirect product.

Let B be any C_2-torus. Then we get that $\pi_1(i_e^* B) \cong \pi_1(T) \cong \mathbb{Z} \times \mathbb{Z} \cong < a, b | aba^{-1}b^{-1} = \text{id} >$. Therefore, $\Pi B(x_*, x_*) =< t, a, b | aba^{-1}b^{-1} = \text{id}, tat = ?, tbt = ?, t^2 = ? >$. The main challenge will be to find out what the question marks are. The rest of the equivariant fundamental groupoid is already given by the fixed set.

We will always denote by $a(e, \cdot)$ the loop at $x_*(e)$ going around the hole of the torus and by $b(e, \cdot)$ the loop at $x_*(e)$ going through the hole. Note that this distinction only makes sense when embedding the torus in $\mathbb{R}^3$. In the intuition of glueing the edges of the torus square, the edge pair which is glued first becomes the loop going around the hole.

We will always paint a in green and b in red. Further, we paint the fixed set in blue and x_* in pink.

2.2.1 The trivial action, T_1^{trivial}

Here we have that every point is a fixed point and $t^2 = \text{id}$, $tat = a$ and $tbt = b$, i.e. everything commutes with t. By these considerations and also directly by Example 1.13, we get the following skeleton of the fundamental groupoid of T_1^{trivial}:

$$\begin{array}{c} \pi_1(T) \\ \circlearrowleft \\ x_1 \\ {\scriptstyle \pi_1(T)} \uparrow \\ x_* \\ \circlearrowright \\ C_2 \times \pi_1(T) \end{array}$$

Here $\Pi T_1^{\text{trivial}}(x_*, x_*) \cong C_2 \times \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T) \rtimes_\varphi C_2$ is the trivial semidirect product, where $\varphi\colon C_2 \to \text{Aut}(\pi_1(T)), \tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts via the unit matrix on $\mathbb{Z} \times \mathbb{Z} \cong \pi_1(T, x_*(e))$.

Thinking of the Torus as a square, i.e. as $T = (\mathbb{R}/\mathbb{Z})^2$, the identity map of $\mathbb{R}^2$ (represented by the identity matrix) factors through $\mathbb{Z}$, which turns it into the trivial action on the torus.

The last two paragraphs might seem redundant, but it is worth noting to give the full picture. It seems that there is a one to one correspondence between self-inverse linear maps in $\mathbb{R}^2$ which factor through to the torus (i.e. similarity classes of self inverse integer matrices) and involutions with fixed points on the torus.

2.2.2 The reflection, $T_1^{\text{refl}}[C = 2]$

We choose $x_* \in \Pi(T_1^{\text{refl}}[C = 2])$ such that $x_*(e) = x_*(\tau)$ is on one of the fixed circles as in Figure 2.1, and we choose t such that $t(e)$ is the constant path at $x_*(e)$.

The C_2-action in the square representation of $T_1^{\text{refl}}[C = 2]$ is given by the reflection along any of the fixed lines (see Figure 2.3). The involution on $T = (\mathbb{R}/\mathbb{Z})^2$ is given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

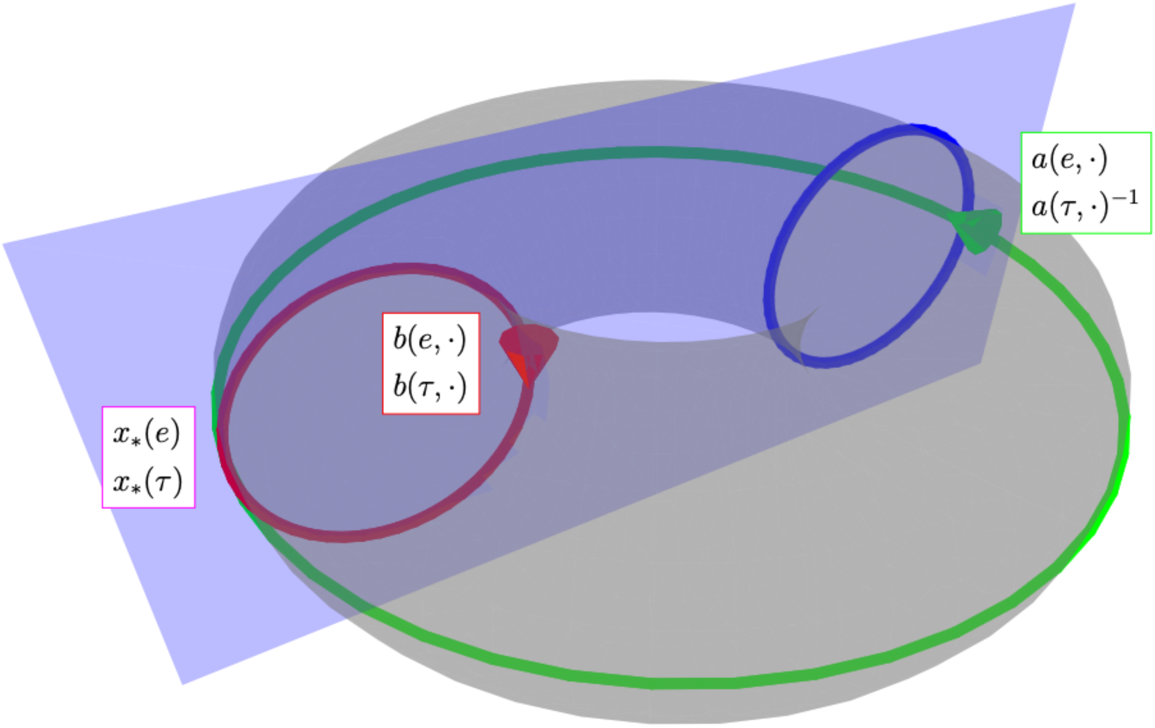


Figure 2.1: $T_1^{\text{refl}}[C=2]$ in $\mathbb{R}^3$

One can see that the reflection tat of a is just a^{-1}, and that b is fixed, i.e. $tbt = b$. Also $t^2 = \text{id}$. This yields $\Pi(T_1^{\text{refl}}[C=2])(x_*, x_*) \cong < t, a, b | aba^{-1}b^{-1} = \text{id}, tat = a^{-1}, tbt = b, t^2 = \text{id} >$, So we get a skeleton of $\Pi(T_1^{\text{refl}}[C=2])$ as given in Figure 2.2.

One can identify $T_1^{\text{refl}}[C=2]$ with $S^{1,1} \times S^1$, the space from Example 1.12 times S^1. This is not only visible in Figure 2.1 and Figure 2.3, but also reflects in the equivariant fundamental groupoid, as $\Pi(T_1^{\text{refl}}[C=2]) \cong (\Pi S^{1,1}) \times \Pi(S^1)$. Especially, $\Pi(T_1^{\text{refl}}[C=2])(x_*, x_*) \cong < t, a, b | aba^{-1}b^{-1} = \text{id}, tat = a^{-1}, tbt = b, t^2 = \text{id} > \cong (\mathbb{Z}\{a\} \rtimes \mathbb{Z}/2\{t\}) \times \mathbb{Z}\{b\} \cong \Pi S^{1,1}(x_*, x_*) \times \Pi S^1(x_*(e), x_*(e))$.

Also $\Pi(T_1^{\text{refl}}[C=2])(x_*, x_*) \cong \pi_1(T) \rtimes_\varphi C_2$ is a semidirect product, where $\varphi\colon C_2 \to \text{Aut}(\pi_1(T)), \tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

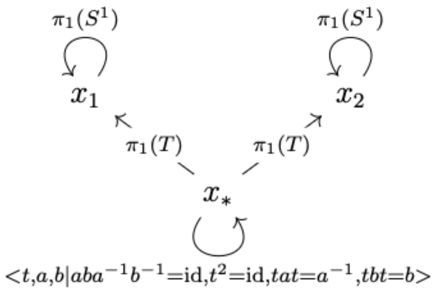


Figure 2.2: $\Pi(T_1^{\text{refl}}[C=2])$

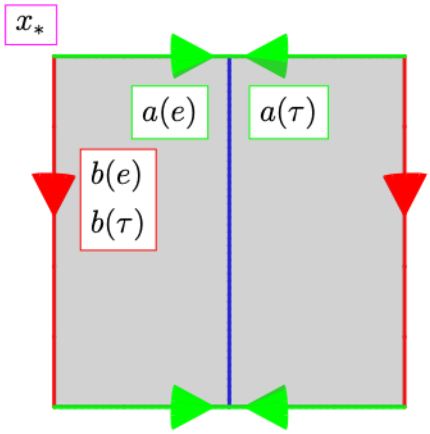


Figure 2.3: $T_1^{\text{refl}}[C=2]$

2.2.3 The spit, $T_1^{\text{spit}}[F=4]$

We choose $x_* \in \Pi(T_1^{\text{spit}}[F=4])$ as a fixed point like in Figure 2.4 and t constant.

The C_2-action in the square representation of $T_1^{\text{spit}}[F=4]$ is given by a 180 degree rotation around any of the fixed points (see Figure 2.6) and the involution on $T=(\mathbb{R}/\mathbb{Z})^2$ is given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

One can see that the image tat of a under the rotation is a^{-1} and the image of b is b^{-1}. Therefore, we get a skeleton of $\Pi(T_1^{\text{spit}}[F=4])$ as given in Figure 2.5.

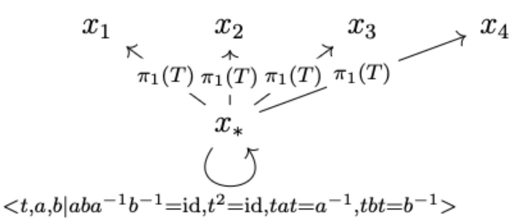


Figure 2.5: $\Pi(T_1^{\text{spit}}[F=4])$

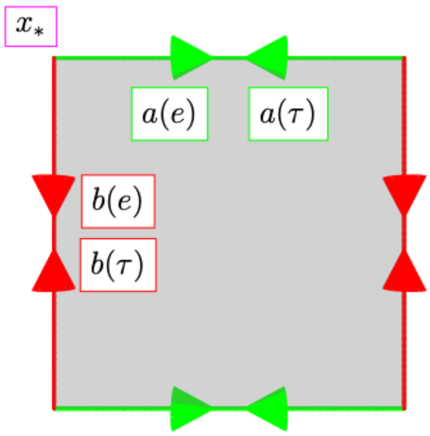


Figure 2.6: $T_1^{\text{spit}}[F=4]$

The rotation in Figure 2.4 can be understood as reflecting along two planes, one containing a and one containing b.

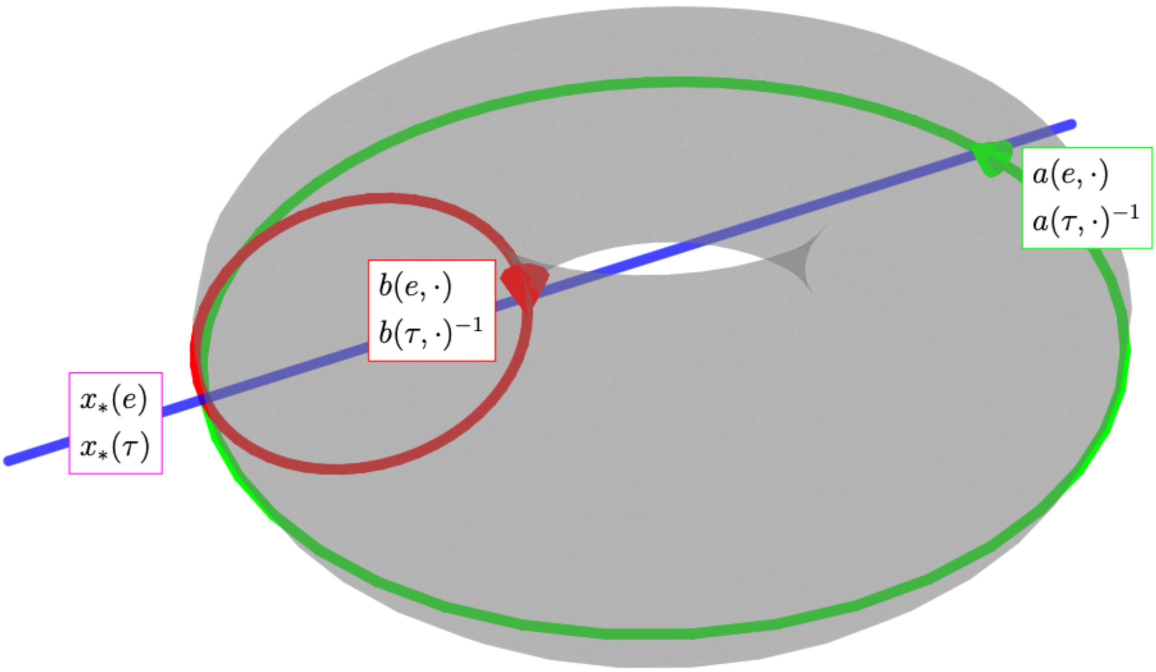


Figure 2.4: $T_1^{\text{spit}}[F=4]$ in $\mathbb{R}^3$

Putting everything together, $\Pi(T_1^{\text{spit}}[F=4])(x_*,x_*) \cong \pi_1(T) \rtimes_\varphi C_2$ is a semidirect product, where $\varphi\colon C_2 \to \operatorname{Aut}(\pi_1(T)), \tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

2.2.4 The antipodal action, $T^{\text{anti}}[1,0]$

Choose $x_*(e)$ on the outer perimeter. Then $x_*(\tau)$ is opposite of $x_*(e)$, and $a(e)$ goes around the outer perimeter once (see Figure 2.7).

Note that $a(\tau)$ also goes around the outer perimeter once, in the same direction as $a(e)$, but rotated by 180° degrees, starting from $x_*(\tau)$. Choose $t(e)$ as the path segment of $a(e)$ from $x_*(e)$ to $x_*(\tau)$. Then $t(\tau)$ is the path segment of $a(e)$ from $x_*(\tau)$ to $x_*(e)$.

Further, note that $b(\tau)$ is in the opposite direction as $b(e)$, and rotated 180° degrees around the torus, starting at $x_*(\tau)$.

The C_2-action on the square representation of $T^{\text{anti}}[1,0]$ (see Figure 2.9) is given by translating by half the square in the a direction, followed by mirroring the b direction along the horizontal middle line or a.

As relations we get $t^2 = a$, $tat = a^2$, $tbt = b^{-1}a$, and a skeleton of $\Pi(T^{\text{anti}}[1,0])$ is given by Figure 2.8

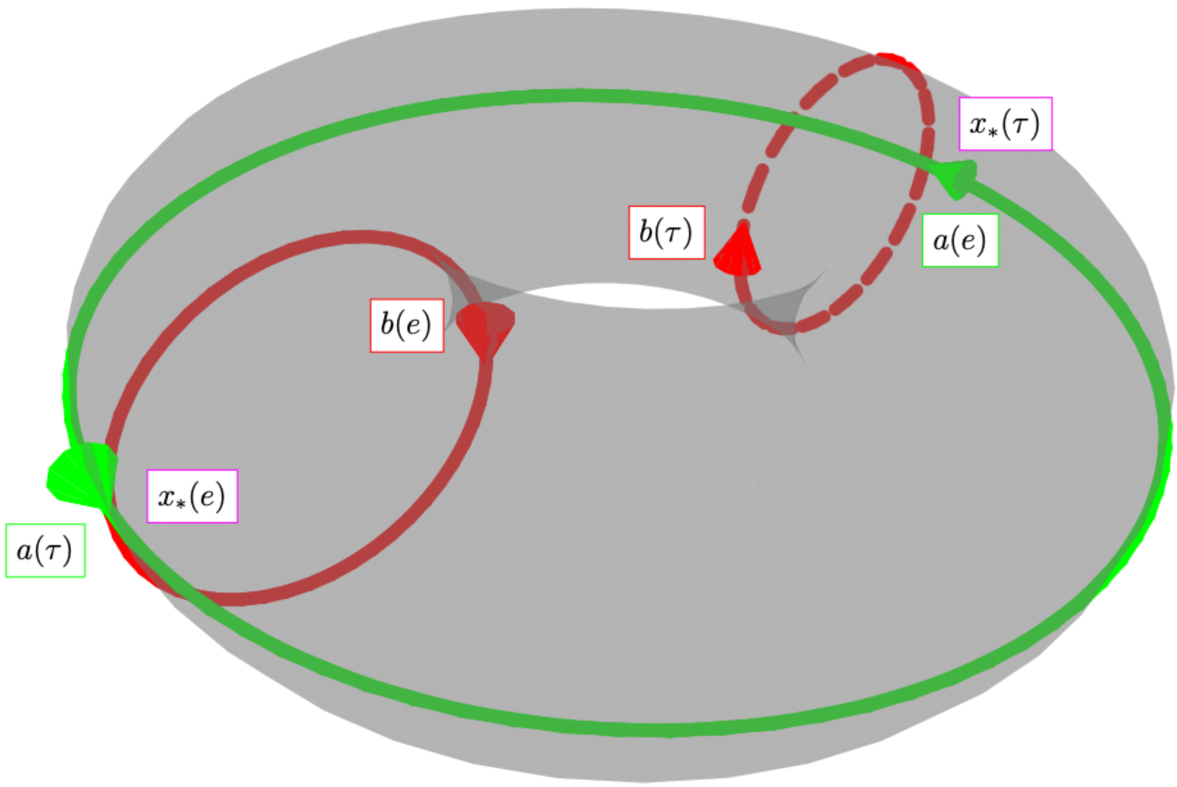


Figure 2.7: $T^{\text{anti}}[1,0]$ in $\mathbb{R}^3$

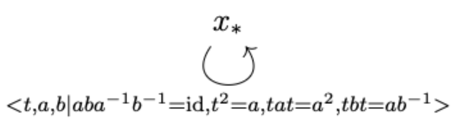


Figure 2.8: $\Pi(T^{\text{anti}}[1,0])$

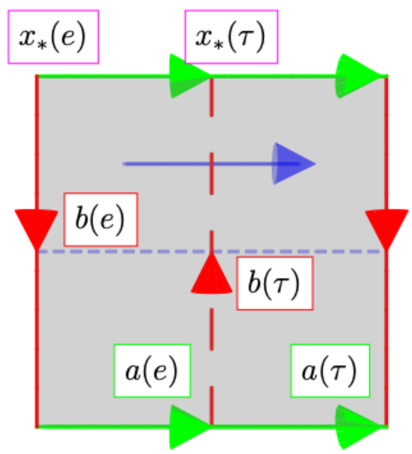


Figure 2.9: $T^{\text{anti}}[1,0]$

Note that since $T^{\text{anti}}[1,0]$ has no fixed points $\Pi(T_1^{\text{anti}}[1,0])(x_*, x_*) \not\cong \pi_1(T) \rtimes_\varphi C_2$ is not a semidirect product of C_2 and $\pi_1(T)$.

However, it is possible to simplify the description of $\Pi(T_1^{\text{anti}}[1,0])(x_*, x_*)$. First notice, that $t^2 = a$, so we can get rid of a as a generator. Then the relation $tat = a^2$ becomes obsolete, the relation $aba^{-1}b^{-1} = \text{id}$ becomes $t^2bt^{-2}b^{-1} = \text{id}$, and the relation $tbt = ab^{-1}$ turns into $tbt = t^2b^{-1} \Leftrightarrow bt = tb^{-1}$.

So far we have $\Pi(T_1^{\text{anti}}[1,0])(x_*, x_*) \cong < t, b | t^2b = bt^2, bt = tb^{-1} >$. Notice that the second relation implies the first, as $bt = tb^{-1} \Leftrightarrow t = b^{-1}tb^{-1} \Leftrightarrow tb = b^{-1}t$ and therefore $t^2b = tb^{-1}t = bt^2$.

So now we have $\Pi(T_1^{\text{anti}}[1,0])(x_*, x_*) \cong < t, b | bt = tb^{-1} >$, which is the non-trivial semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$ and the fundamental group of the Klein bottle. Taking a step back from the calculation and looking at a, b, and t in the pictures of $T_1^{\text{anti}}[1,0]$, this also makes intuitive sense as the quotient space $T_1^{\text{anti}}[1,0]/C_2$ is the Klein Bottle: Note that the quotient space looks like half the torus square, which is the rectangle bounded by t and b with identifications $bt = tb^{-1}$.

2.2.5 The free rotation, T_1^{rot}

Again, choose $x_*(e)$ on the outer perimeter. Then $x_*(\tau)$ is opposite of $x_*(e)$, and $a(e)$ goes around the outer perimeter once (see Figure 2.10).

As with $T^{\text{anti}}[1,0]$, $a(\tau)$ is in the same direction as $a(e)$ and we choose $t(e)$ to be the first half of $a(e)$, so $t(\tau)$ is the second half.

Unlike in the previous C_2-Torus, here $b(\tau)$ is in the same direction as $b(e)$.

The C_2-action on the square representation of T_1^{rot} is given by translation by half a square in the a direction (see Figure 2.12). As relations we get $t^2 = a$, $tat = a^2$, $tbt = ab$, and a skeleton of $\Pi(T_1^{\text{rot}})$ is given by Figure 2.11:

Note that since T_1^{rot} has no fixed points, $\Pi(T_1^{\text{rot}})(x_*, x_*) \not\cong \pi_1(T) \rtimes_\varphi C_2$ is not a semidirect product of C_2 and $\pi_1(T)$.

However, there is again a simplified way to write down $\Pi(T_1^{\text{rot}})(x_*, x_*)$. Because of $t^2 = a$, we can get rid of a as a generator again. The relation $tat = a^2$ becomes trivial, the relation $aba^{-1}b^{-1} = \text{id}$ becomes $t^2bt^{-2}b^{-1} = \text{id}$, and the relation $tbt = ab$ turns into $tbt = t^2b \Leftrightarrow bt = tb$.

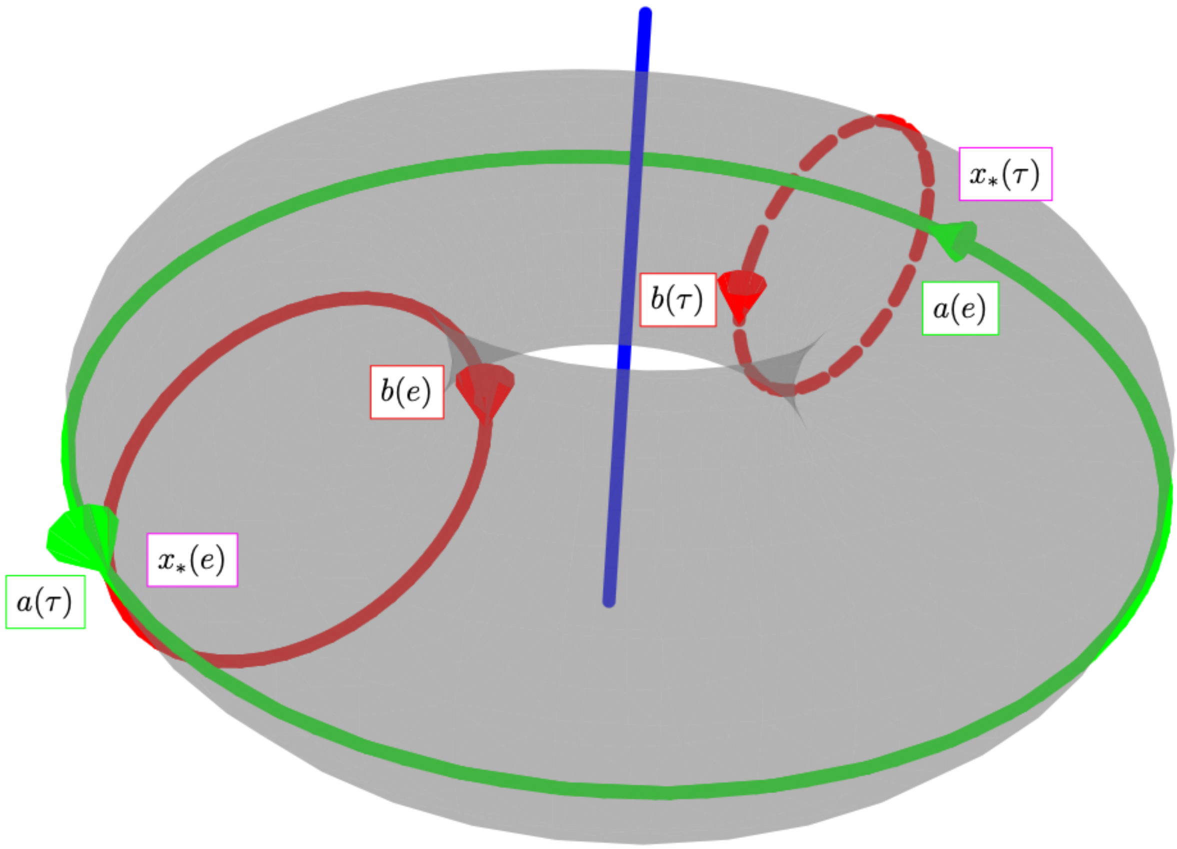


Figure 2.10: T_1^{rot} in $\mathbb{R}^3$

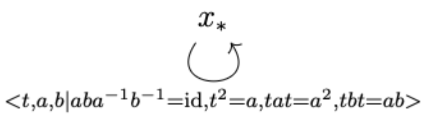


Figure 2.11: $\Pi(T_1^{\text{rot}})$

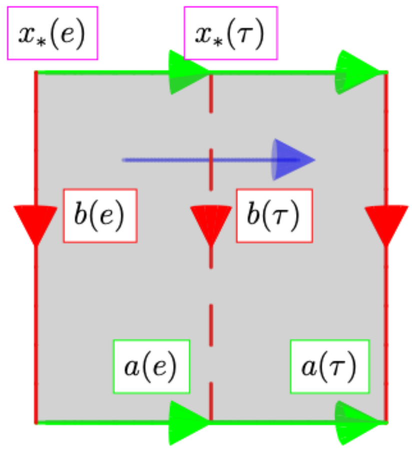


Figure 2.12: T_1^{rot}

So far we have that $\Pi(T_1^{\mathrm{rot}})(x_*, x_*) \cong < t, b | t^2 b = bt^2, bt = tb >$. Again, the first relation becomes obsolete by the second, and we obtain that $\Pi(T_1^{\mathrm{rot}})(x_*, x_*) \cong < t, b | bt = tb > \cong \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T)$. Again this also makes intuitive sense in both pictures, as the quotient space T_1^{rot}/C_2 is the torus, of which t and b generate its fundamental group.

Especially one gets that $\Pi(T_1^{\mathrm{rot}})(x_*, x_*) \not\cong \Pi(T_1^{\mathrm{anti}}[1,0])(x_*, x_*)$, so the skeletons of $\Pi(T_1^{\mathrm{rot}})$ and $\Pi(T_1^{\mathrm{anti}}[1,0])$ do not agree and therefore they are not equivalent categories.

2.2.6 The toilet paper roll, $T^{\mathrm{anti}}[0,1]$

Recall the general construction of $T^{\mathrm{anti}}[\cdot,\cdot]$. In this special case, we start with S^2_a, the sphere with the antipodal action, then cut out an open disk which is disjoint from its image under the antipodal action and also cut out the image disk. Then we identify the boundaries of the disks, such that it becomes a fixed circle.

The sphere with antipodal disks cut out is homeomorphic to the cylinder, and also the action is still antipodal on the cylinder. Glueing the boundaries creates a torus. We choose x_* on the boundary of the cylinder which is fixed, t constant, $b(e)$ to go around the boundary of the cylinder once, and $a(e)$ as in Figure 2.13.

The C_2-action on the square representation of $T^{\mathrm{anti}}[0,1]$ is best understood by skewing the square, in which case the C_2-action is given by reflection along b followed by a half-shift in the b-direction. In an unscewed square, i.e. in the $T = (\mathbb{R}/\mathbb{Z})^2$ picture, the action is a skew followed by a reflection. It is given by the matrix $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

We get the identifications $t^2 = \mathrm{id}$, $tbt = b$, $tat = ba^{-1}$, which determine the following skeleton of the toilet paper roll:

Since the toilet paper roll has fixed points, $\Pi(T^{\mathrm{anti}}[0,1])(x_*, x_*) = \pi_1(T) \rtimes_\varphi C_2$ is a semidirect product, and $\varphi\colon C_2 \to \mathrm{Aut}(\pi_1(T)), \tau \mapsto \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

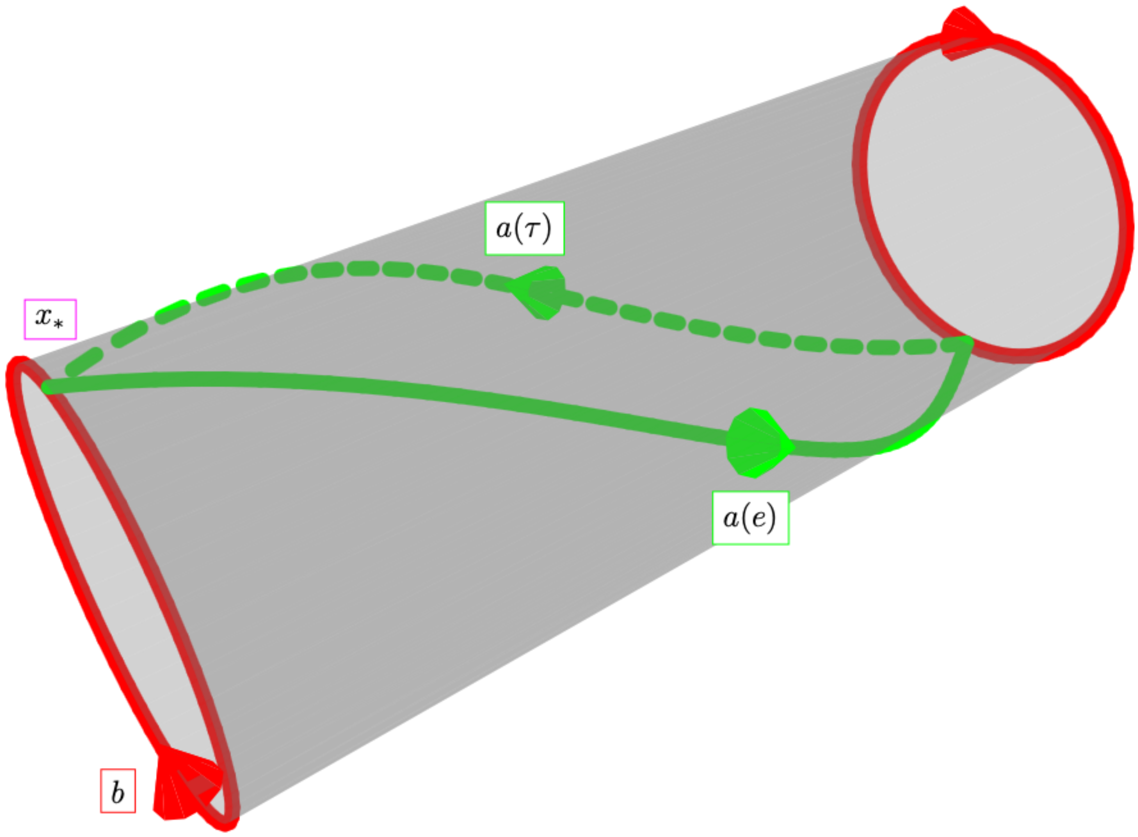


Figure 2.13: $T^{\text{anti}}[0, 1]$ in $\mathbb{R}^3$

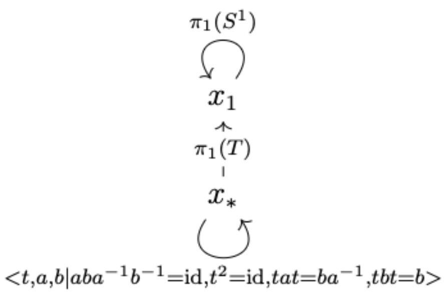


Figure 2.14: $\Pi(T^{\text{anti}}[0, 1])$

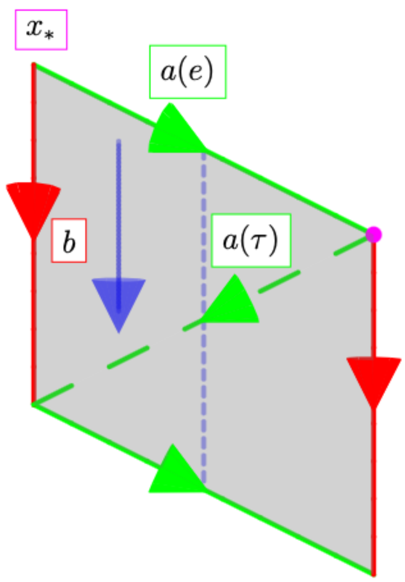


Figure 2.15: $T^{\text{anti}}[0, 1]$

2.2.7 Conclusions for all C_2-Tori

Let's try to answer the open questions 1.36 and 1.37 for the torus. They both turn out to be true:

Theorem 2.2. *Let B be a C_2-torus. Then ΠB determines which one.*

Proof. All skeletons of equivariant fundamental groupoids of C_2-tori above are different from each other. □

Theorem 2.3. *Let H be a semidirect product of the form $H := \pi_1(T) \rtimes_\varphi C_2$. Then there exists a C_2-torus with $\Pi B(x_*, x_*) \cong H$.*

Proof. A semidirect product is given by $\varphi\colon C_2 \to \mathrm{Aut}(\pi_1(T)) \cong \mathbb{Z}^{2\times 2}$, which in turn is determined by $\varphi(\tau)$, which is a 2×2 self-inverse integer matrix.

Notice further, that the linear map $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by a self-inverse integer matrix $\varphi(\tau)$ factors through to an involution $\tilde{\phi}\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ on the torus with fixed point $x_* := \mathbb{Z}^2$ by the universal property of the quotient. Let B be that C_2-surface. The paths a, b along the unit vectors, are generators of $\pi_1(i_e^* B, x_*)$. Notice that $\phi(a)$ is the straight line from the origin to $\varphi(\tau)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\varphi(\tau)_{1,1}\\ \varphi(\tau)_{2,1}\end{pmatrix}$, and $\phi(b)$ is the straight line from the origin to $\varphi(\tau)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\varphi(\tau)_{1,2}\\ \varphi(\tau)_{2,2}\end{pmatrix}$.

These straight lines are path homotopic to the paths along the grid lines, and since the quotient $\rho\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is the universal covering of the torus, we get that $[\rho(\phi(a))] = [a^{\varphi(\tau)_{1,1}} b^{\varphi(\tau)_{2,1}}]$ in $\pi_1(i_e^* B, x_*)$ and $[\rho(\phi(b))] = [a^{\varphi(\tau)_{1,2}} b^{\varphi(\tau)_{2,2}}]$.

Then $\Pi B(x_*, x_*) \cong < t, a, b | aba^{-1}b^{-1} = \mathrm{id}, t^2 = \mathrm{id}, tat = a^{\varphi(\tau)_{1,1}} b^{\varphi(\tau)_{2,1}}, tbt = a^{\varphi(\tau)_{1,2}} b^{\varphi(\tau)_{2,2}} > \cong \pi_1(T) \rtimes_\varphi C_2$. □

2.3 Free C_2-Klein bottle

Let $K = N_2$ be the connected sum of two $\mathbb{R}P^2$, known as the Klein-bottle. We will calculate the equivariant fundamental groupoid of its unique free involution. For a model of K in $\mathbb{R}^3$ one can think of the quotient space of the torus with antipodal action, $K = T^{\mathrm{anti}}[0,1]/C_2$ (See Section 2.2.4). As square model we take the usual Klein Bottle square with boundary word $aba^{-1}b$ as in Figure 2.16. Then a (and by Dugger [Dug16] the unique) free action on the Klein bottle is a half shift up/down in direction of b.

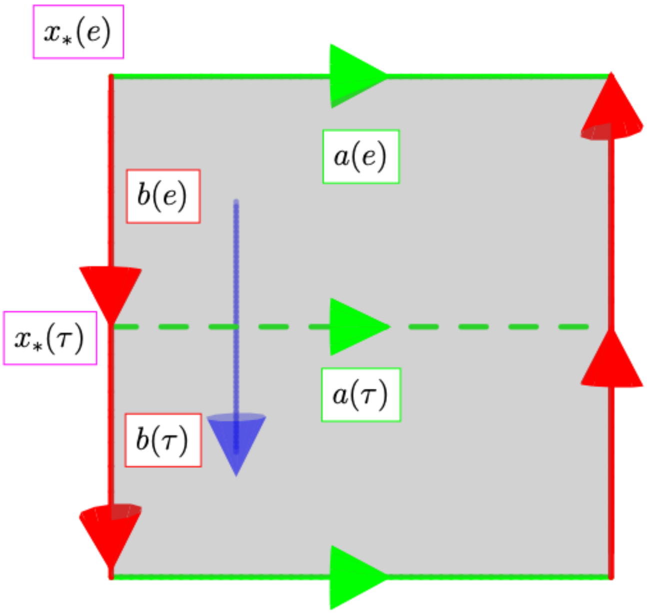


Figure 2.16: K^{free}

Let's denote this C_2-surface by K^{free}, and choose the t as the first half of b. Then $t^2 = b$.

One can see that the quotient space K^{free}/C_2 is given by half the square above, with edge identifications $ata^{-1}t$, i.e. $K^{\text{free}}/C_2 \cong K$ is again the Klein bottle. Therefore we get that $\Pi K^{\text{free}}(x_*, x_*) \cong \pi_1(K) \cong \mathbb{Z} \rtimes \mathbb{Z}$, and ΠK^{free} has the following as a skeleton:

This is exactly the same skeleton as for $\Pi(T^{\text{anti}}[0, 1])$.

This shows that in general, given an equivariant fundamental groupoid of a surface, one cannot infer the underlying surface. This is unlike the non-equivariant situation where the fundamental groupoid differentiates all closed surfaces.

One can ask if given the underlying space or its fundamental groupoid and the equivariant fundamental groupoid, one can infer the C_2-surface. This is Question 1.36.

2.4 The $T_{g,r}^{\text{refl}}$

Let's choose the generators of $\pi_1(T_{g,r}^{\text{refl}})$ as in Figure 2.17, which shows the front side of the T_g embedded in $\mathbb{R}^3$).

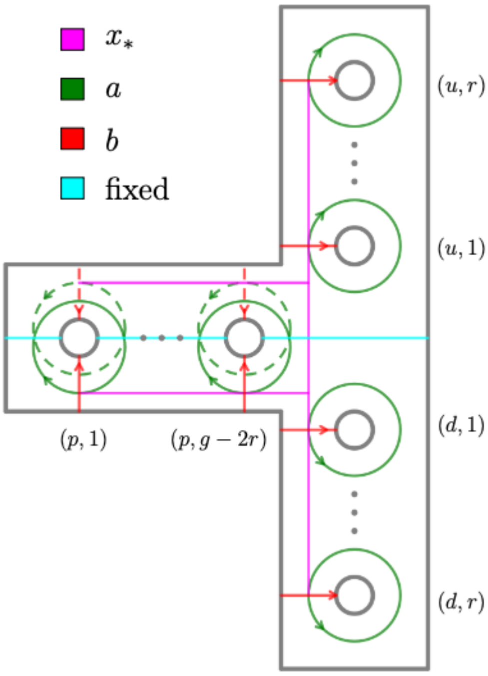


Figure 2.17: $T_{g,r}^{\text{refl}}$ in $\mathbb{R}^3$

To understand this picture, it helps to compare it with Figure 1.4. It is the same picture but the viewing-angle is more from the front and the torus is less smooth.

The purple part is contractible and mapped to itself under the reflection, and therefore it is possible for us to consider it as one identified fixed point x_*. For the Loops on the left, the dashed loops are the images of the non-dashed loops under the reflection.

One can immediately see, how the generators belonging to the $2r$ holes on the right are mapped onto each other. The difficulty for writing down the relations is describing the dotted loops as a composition of the generating loops. Also, it is not at first obvious, what the polygon representation is, and therefore what the identification of the fundamental group is. For now let's just accept the polygon representation in Figure 2.18:

Again we help ourselves by not considering x_* as only the corners (i.e. the path start- and endpoints) but we extend one of the corners by the pink curve as in the picture.

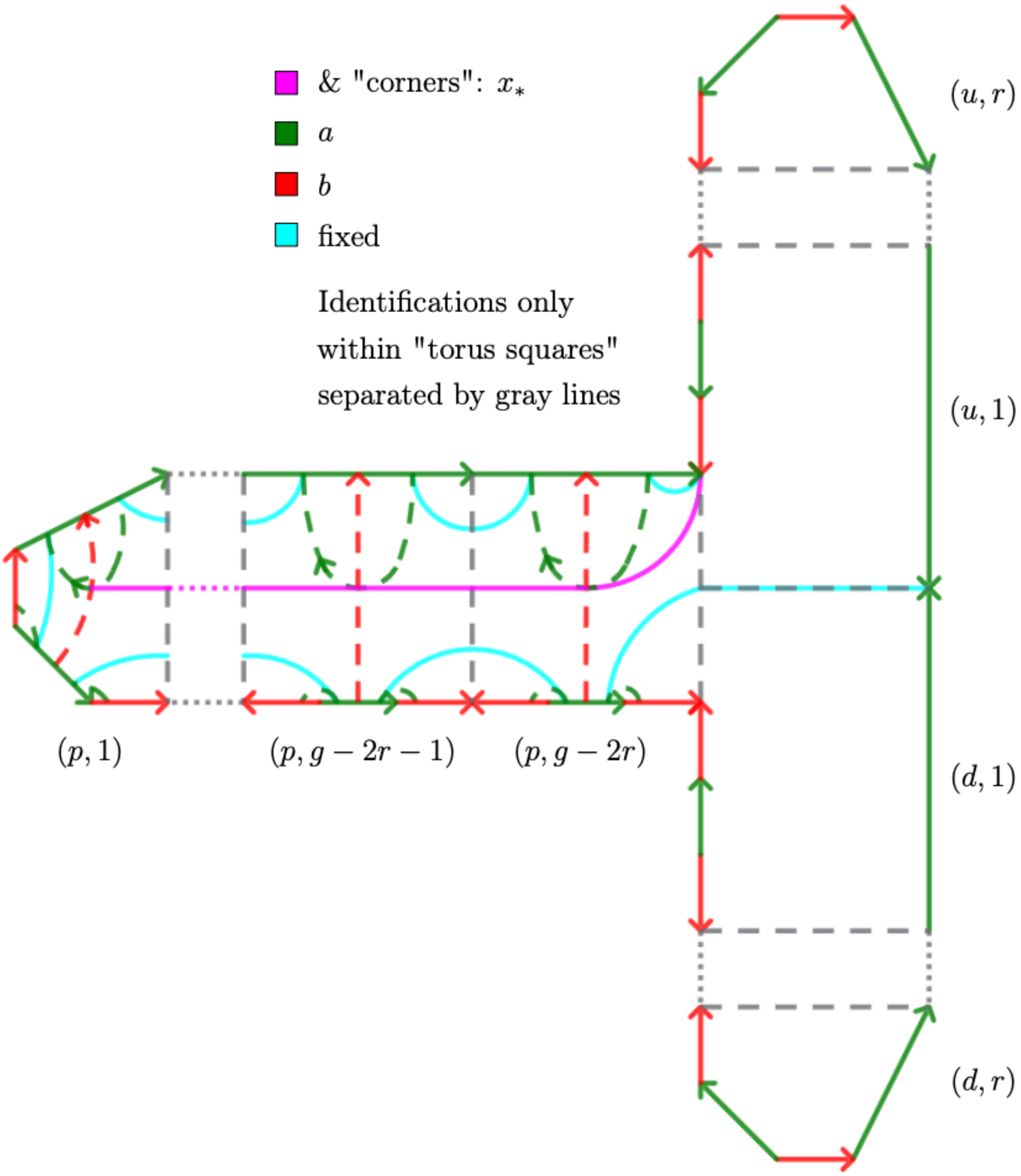


Figure 2.18: $T_{g,r}^{\text{refl}}$ as $4g$-gon

The dashed gray lines divide the polygon into connected "torus squares". Each of those corresponds to the surface around one of the torus holes, and identifications of the arrows only happens within those.

We first notice that the order in which all the curves intersect matches in Figure 2.17 and Figure 2.18, which supports the claim, that the polygon representation is valid. The involution is hard to grasp in the polygon representation, however it still is something like "reflection along the fixed set".

The identification of the fundamental group is given by walking around the boundary of it once.

For talking about the relations, we need to introduce variable names for the loops.

We first assign labels to the different types of holes. We call the ones intersecting the reflection plane by p, those above by u (up), and those below by d (down). Now we enumerate the holes on the reflection plane from left to right by $(p, 1)$ to $(p, g - 2r)$. For the holes above the reflection line we choose the labels $(u, 1), \dots, (u, r)$ from the middle to the outside, and for the holes below we choose the labels $(d, 1), \dots, (d, r)$, also from the middle to the outside.

Then denote all the green loops by $a_{(\cdot,\cdot)}$, and all the red loops by $b_{(\cdot,\cdot)}$. Regarded as objects of $\Pi T_{g,r}^{\text{refl}}$, we get that $a_{(p,\cdot)}(\tau)$ are the dashed green loops and $b_{(p,\cdot)}(\tau)$ are the dashed red loops. The red loops always go through the hole, and the green loops always go around the hole. Let's also define the loops $a'_{(\cdot,\cdot)} \coloneqq b_{(\cdot,\cdot)}^{-1} a_{(\cdot,\cdot)} b_{(\cdot,\cdot)}$. In the polygon-representation these correspond to the 3 connected edges in each area corresponding to one of the torus holes. In the embedded representation they correspond to loops in the same direction as the corresponding a, but on the back side. The conjugation by the corresponding b "pulls a through the hole to the back side". Last, choose (t, τ) such that t is the constant paths at x_*.

With this intuition, we can now understand all identifications in $\Pi T_{g,r}^{\text{refl}}(x_*, x_*)$ coming from the loops along the reflection plane.

First notice that for the right most hole on the reflection plane we have $t a_{(p,g-2r)} t = a_{(p,g-2r)}^{-1}$. Further notice, that for the second to right hole on the reflection plane, we almost have that $t a_{(p,g-2r-1)} t$ equals $a_{(p,g-2r-1)}^{-1}$. The only difference is if the start and end segments of the loop are above or below the hole $(p, g - 2r)$ at the right. This difference is exactly given by $a_{(p,g-2r)}$, and therefore $t a_{(p,g-2r-1)} t = a_{(p,g-2r)}^{-1} a_{(p,g-2r-1)}^{-1} a_{(p,g-2r)}$.

In general for $ta_{(p,\cdot)}t$ we therefore get

$$ta_{(p,i)}t = (a^{-1}_{(p,g-2r)} \cdots a^{-1}_{(p,i+1)})a^{-1}_{(p,i)}(a_{(p,i+1)} \cdots a_{(p,g-2r)})$$

For seeing this in the polygon representation, remember that the base point of the dashed green loops $ta_{(p,i)}t$ can be moved along the purple line to the corner of the polygon to which the purple line is connected to. Then notice, that the first half of the loop is path-homotopic to walking the border segment $(a^{-1}_{(p,g-2r)} \cdots a^{-1}_{(p,i+1)})a^{-1}_{(p,i)}$ while the second half is path-homotopic to the border segment $(a_{(p,i+1)} \cdots a_{(p,g-2r)})$.

The red dashed loops $tb_{(p,\cdot)}t$ are a bit harder to imagine as composition in the embedded picture, so let's start with the polygon. If we again move the base point of a loop $tb_{(p,i)}t$ along the purple line, we find that the first half of the loop is again path-homotopic to the border segment $(a^{-1}_{(p,g-2r)} \cdots a^{-1}_{(p,i+1)})a^{-1}_{(p,i)}$. We choose to include $a^{-1}_{(p,i)}$ in the first half, but it could have been put in the second half as well. The second half is path-homotopic to the border segment $b_{(p,i)}(a'^{-1}_{(p,i-1)} \cdots a'^{-1}_{(p,1)})(a_{(p,1)} \cdots a_{(p,g-2r)})$. So in total we find that

$$tb_{(p,\cdot)}t = (a^{-1}_{(p,g-2r)} \cdots a^{-1}_{(p,i)})b_{(p,i)}(a'^{-1}_{(p,i-1)} \cdots a'^{-1}_{(p,1)})(a_{(p,1)} \cdots a_{(p,g-2r)})$$

.

An interpretation in the embedded picture is the following: $(a^{-1}_{(p,g-2r)} \cdots a^{-1}_{(p,i)})$ represents the loop going around the holes $(p, g-2r), \ldots, (p, i)$ anticlockwise on the front. This loop we compose with the loop $b_{(p,i)}$, which is going through the hole (p, i) in the direction needed for the red dashed loop $tb_{(p,i)}t$. Together, the loop can be imagined as coming from the purple line above the (p, i)-hole, passing through it at the left side of it, continuing downwards on the back, around the bottom and arriving at the bottom purple line in the front.

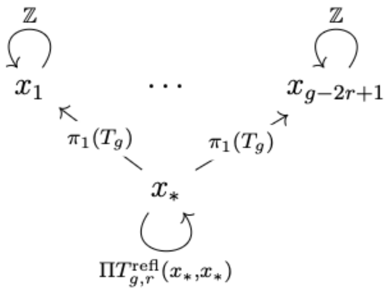
This loop we now compose with $(a'^{-1}_{(p,i-1)} \cdots a'^{-1}_{(p,1)})$, which is the loop going around the holes $(p, i-1), \ldots, (p, 1)$ anticlockwise on the back side. The composed loop so far can now be imagined as a loop around all $(p, \cdot)$-holes, switching from front to back in the (p, i)-hole and arriving at the front again on the bottom.

Now we just have to walk around all $(p, \cdot)$-holes again to only be left with passing through the (p, i)-hole. This is done by composing with $(a_{(p,1)} \cdots a_{(p,g-2r)})$.

Similarly, one can also try to make sense of the identification of the fundamental group in the embedded case, which is explicitly given by

$$(a'^{-1}_{(p,g-2r)}\cdots a'^{-1}_{(p,1)})(a_{(p,1)}\cdots a_{(p,g-2r)})$$
$$(a'^{-1}_{(u,1)}\cdots a'^{-1}_{(u,r)})(a_{(u,r)}\cdots a_{(u,1)})$$
$$(a^{-1}_{(d,1)}\cdots a^{-1}_{(d,r)})(a'_{(u,r)}\cdots a'_{(u,1)})$$

Now that we have figured out all identifications, we can give the fundamental groupoid $\Pi T_{g,r}^{\text{refl}}$:



where

$$\begin{aligned}
\Pi T_{g,r}^{\text{refl}}(x_*,x_*) = <\,& a_{(p,1)}, b_{(p,1)}, \cdots, a_{(p,g-2r)}, b_{(p,g-2r)},\\
& a_{(u,1)}, b_{(u,1)}, \cdots, a_{(u,r)}, b_{(u,r)},\\
& a_{(d,1)}, b_{(d,1)}, \cdots, a_{(d,r)}, b_{(d,r)},\\
& t\\
\mid\ & t^2 = \text{id},\\
& ta_{(d,i)}t = a_{(u,i)}, tb_{(d,i)}t = b_{(u,i)} && \forall i : 1 \leq i \leq r,\\
& ta_{(u,i)}t = a_{(d,i)}, tb_{(u,i)}t = b_{(d,i)} && \forall i : 1 \leq i \leq r,\\
& ta_{(p,i)}t = (a^{-1}_{(p,g-2r)}\cdots a^{-1}_{(p,i+1)})a^{-1}_{(p,i)}(a_{(p,i+1)}\cdots a_{(p,g-2r)}) && \forall i : 1 \leq i \leq g-2r,\\
& tb_{(p,i)}t = (a^{-1}_{(p,g-2r)}\cdots a^{-1}_{(p,i)})b_{(p,i)}\\
& \qquad (a'^{-1}_{(p,i-1)}\cdots a'^{-1}_{(p,1)})(a_{(p,1)}\cdots a_{(p,g-2r)}) && \forall i : 1 \leq i \leq g-2r,\\
& \text{id} = (a'^{-1}_{(p,g-2r)}\cdots a'^{-1}_{(p,1)})(a_{(p,1)}\cdots a_{(p,g-2r)})\\
& \quad (a'^{-1}_{(u,1)}\cdots a'^{-1}_{(u,r)})(a_{(u,r)}\cdots a_{(u,1)})\\
& \quad (a^{-1}_{(d,1)}\cdots a^{-1}_{(d,r)})(a'_{(u,r)}\cdots a'_{(u,1)})\ >
\end{aligned}$$

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